One dimensional algebraic geometry

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Declaration

The work in this thesis is my own except where otherwise stated.

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Abstract

In this thesis, we give a geometric exposition of one dimensional regular schemes, which we call Dedekind schemes. We develop the algebro-geometric language needed to discuss these objects, along with some of the less elementary commutative algebra required. We use completion to analyse the local invariants of morphisms of Dedekind schemes, and show how the trace map describes the different and the discriminant divisors associated to such a morphism. We develop the theory of quasicoherent sheaves on Dedekind schemes, in order to investigate some of their global invariants, such as the Picard group, the Class group, and their categories of coherent sheaves. We end with a treatment of curves over a field, and a proof of the Riemann-Roch theorem using adeles and Weil differentials.

Contents

Notation

Notation

O_X	The structure sheaf of a locally ringed space or scheme.
$O_{X,P}$	The stalk of the structure sheaf O_X at a point P.
$\mathcal{F}_P$	The stalk of a sheaf $\mathcal{F}$ at a point P.
$K(X)$	The fraction field of an integral scheme X, definition 2.46.
$\mathrm{Quot}(R)$	The fraction field of an integral domain R.
R^*	The group of units of R, when R is a ring.
V^*	The dual vector space of V, when V is a vector space over k.
Q/P	With reference to a map $f : X \to Y$, points $Q \in X$, $P \in Y$ such that $f(Q) = P$.
$R_\mathfrak{p}$	The localisation of a ring R at a prime ideal $\mathfrak{p}$.
$\widehat{M}^I$	The I-adic completion of an R-module M with respect to the ideal $I \subset R$.
$\widehat{O}^{\mathfrak{p}}_{X,P}$	The $\mathfrak{p}$-adic completion of $O_{X,P}$, the stalk of the structure sheaf O_X at a point P.
$\widehat{K(X)}^P$	The field of fractions of $\widehat{O}^{\mathfrak{p}}_{X,P}$.
$\mathbb{A}_X$	The adele ring associated to a curve X.
$\mathrm{Pic}(X)$	The Picard group of invertible sheaves on X, definition 4.33.
$\mathrm{Cl}(X)$	The class group of X, definition 3.42.
L_D	The global sections of the sheaf $\tilde{L}(D)$, definition 5.21.

Chapter 1

Introduction

Consider the ring $\mathbb{Z}[i]$, and the prime ideals (2), (3) and (5) in $\mathbb{Z}$. In $\mathbb{Z}[i]$, these ideals decompose into products of prime ideals as follows.

$$(2) = (1+i)^2$$
$$(3) = (3)$$
$$(5) = (2+i)(2-i)$$

When k is a field, we see similar algebraic phenomena occuring between prime ideals of $k[t]$ and $k[x]$ under the following ring homomorphism.

$$k[t] \to k[x]$$
$$t \to x^2$$

Now letting $k = \mathbb{C}$, we may realise these polynomials as complex valued functions on the topological space $\mathbb{C}$. We also have a bijection between points of this space $\mathbb{C}$ and nonzero prime ideals of $\mathbb{C}[t]$ given by

$$z \longleftrightarrow (t-z).$$

Translating the algebraic behaviour of prime ideals along this bijection yields an associated two sheeted ramified covering map

$$\mathbb{C} \to \mathbb{C}.$$

Our aim in this thesis is to convince the reader that this geometric perspective can be applied to our first example too, and that it can be genuinely useful to do so.

The algebraic phenomena of prime ideals decomposing like this occurs in a special class of rings, Dedekind domains. To any such ring R, we will construct a corresponding space $\mathrm{Spec}(R)$ which realises R as the "functions on $\mathrm{Spec}(R)$". The geometric objects we will consider in this thesis will be spaces that locally look $\mathrm{Spec}(R)$, for R a Dedekind domain. For this, we will use the modern language of algebraic geometry, that of schemes and sheaves. This will allow for a uniform and geometric treatment of these objects, as they are all one dimensional regular schemes, which we call Dedekind schemes.

By treating these objects uniformly, one may clearly identify the formal geometric phenomena that result from this framework. We believe that this geometric approach simplifies and illuminates many aspects of the number theoretic situation.

In our second chapter, we will develop the language of algebraic geometry that we will be needed to discuss these objects. We will also discuss the algebraic properties of Dedekind domains, and introduce discrete valuations as a method of understanding them. The reader familiar with algebraic geometry and commutative algebra need only read our definition 2.56 of a Dedekind scheme and the equivalent description of Theorem 2.62 in order to follow the later chapters.

Our third chapter concerns finite morphisms of Dedekind schemes. We will extensively use the algebraic concepts of integrality and completion, for the latter we will provide a self contained introduction. It is in this setting that we investigate the ramification phenomena in morphisms of Dedekind schemes. We will also introduce the trace form associated to a finite morphism. We will show how this trace completely controls the ramification phenomena, in terms of the different and discriminant divisors. The reader familiar with the first third of [Ser79] should feel comfortable with the content of this chapter.

In chapter four, we introduce the category of coherent sheaves on our Dedekind schemes, the direct generalisation of finitely generated R modules over a ring R. For a finite morphism of Dedekind schemes, we construct the pushforward and pullback functors between their respective categories of coherent sheaves. Via descent, we also construct the exceptional right adjoint to the pushforward functor. For our Dedekind schemes, we introduce the Picard group, and prove that it is canonically isomorphic to the class group.

We introduce the Grothendieck group of the category of coherent sheaves, then for a Dedekind scheme X, construct the chern isomorphism

$$K_0(\mathrm{Coh}(X)) \to \mathbb{Z} \oplus \mathrm{Pic}(X).$$

We round out the chapter with an interpretation of the different and discriminant divisors in this framework. This chapter is undoubtedly the most technical, though it should be reasonably easy reading for the reader well acquainted with quasicoherent sheaves.

Finally, in chapter five we consider complete nonsingular curves over fields, and enjoy the bountiful structure given by the presence of a base field. After proving the equivalence with function fields, we introduce the ring of adeles associated to a curve. Taking this adelic approach allows for a rapid proof of the Riemann-Roch Theorem for algebraic curves over arbitrary base field. As a corollary, we obtain the Riemann-Hurwitz formula for separable finite morphisms of curves. We finish the chapter by illustrating the relation between Weil differentials, and regular differentials on a curve.

None of the results or ideas in this thesis are new. This thesis owes the largest share of intellectual debt to Dino Lorenzini's fantastic book, "An Introduction to Arithmetic Geometry"[Lor96]. Serre's books [Ser79], and [Ser88] also played a significant role in the author's understanding of this material. We also owe a significant debt to [Har77], in which most of the proofs of unproven statements may be found. Our proof of the Riemann Roch Theorem is essentially due to Andre Weil [Wei74], building on [Sch31].

We will take basic category theory as assumed knowledge, along with a basic understanding of commutative algebra and topology. The reader familiar with the first half of Maclane's text [Mac98] and Atiyah and Macdonald's classic [AM69] should be well equipped to read this thesis.

Chapter 2

Sheaves and schemes

2.1 Sheaves

For a topological space X, we define $Op(X)$ to be the partially ordered set of open subsets of X, viewed as a category. We will use this category as an index set for the data we will associate to X.

Definition 2.1 (Presheaves)**.** Let C be a category. A C valued presheaf $\mathcal{F}$ on X is a functor $\mathcal{F} : Op(X)^{op} \to C$. These form a category $\mathrm{PSh}_C(X)$, with morphisms natural transformations of functors.

Explicitly, this is is an assignment of an object $\mathcal{F}(U)$ of C to each open set of X, along with compatible restriction maps $r^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$, for each inclusion of open sets $V \subset U$.

This is a way to organise data on X, but we want presheaves of a special form, which allow us to glue local data into global data.

Definition 2.2 (Sheaves)**.** Let C be a category with finite limits. A C valued sheaf on X is a C valued presheaf $\mathcal{F}$ such that if $\{U_i\}_{i\in I}$ is an open cover of U open in X, then the following diagram is an equaliser.

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

The double arrows are the products of the restriction maps associated to $U_i \cap U_j \subset U_i$ and $U_i \cap U_j \subset U_j$.

The category $\mathrm{Sh}_C(X)$ of C valued sheaves is the full subcategory of $\mathrm{PSh}_C(X)$ with objects the C valued sheaves on X.

When the category C is clear from context, we will drop the reference to it, and refer to these categories as $\mathrm{Sh}(X)$ and $\mathrm{PSh}(X)$.

A presheaf $\mathcal{F}$ on X is an assignment of compatible local data to each open subset of X, and the sheaf condition gives a "patching" property, we may patch together compatible local data along a cover of U to get a unique piece of data defined on U.

Example 2.3 (Continuous functions)**.** For a topological space X, we can upgrade the ring of continuous real valued functions $C[X, \mathbb{R}]$ into a sheaf C_X in a natural way, by setting $C_X(U) = C[U, \mathbb{R}]$, with the restriction maps given by restriction of functions.

To see that this is a sheaf, we just need to check that if we have $\{f_i\}_{i \in I}$ a family of continuous functions defined on $\{U_i\}_{i \in I}$ an open cover of U, such that $f_i|_{U_j \cap U_i} = f_j|_{U_j \cap U_i}$, then there is a unique function f on U that restricts to f_i on each U_i. For this, we define $f(x) = f_i(x)$ for $x \in U_i$, and our conditions give that this function is continuous, restricts to each f_j on U_j, and is clearly unique.

This is in fact a sheaf of rings, and this prototypical example should be kept in mind for the sheaves of rings we will be encountering throughout this thesis.

Example 2.4 (Sheaves of sections)**.** Another more geometric example is that of sections of a map of topological spaces $f : X \to Y$. This is a sheaf of sets S_f on Y, with value on an open set U given by the sections of f over U.

$$S_f(U) = \{g : U \to X | f \circ g = \mathrm{id}_U\}.$$

The restriction maps are given by restriction of sections, and one can check that sections satisfy the sheaf condition, by a similar argument to the previous example.

Remark 2.5. We may recognise our first example as being the sheaf of sections of the projection $X \times \mathbb{R} \xrightarrow{\pi} X$.

For an arbitrary sheaf $\mathcal{F}$ with values in a concrete category C, we will refer to the elements of $\mathcal{F}(U)$ as sections over U, and sections over X as global sections.

Given a map of topological spaces $f : X \to Y$, we have two primary functors between their categories of presheaves, the direct image and the inverse image.

Definition 2.6 (Pushforward)**.** For a presheaf $\mathcal{F}$ on X, define the pushforward $f_*\mathcal{F}$ of $\mathcal{F}$ by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$, with the restriction maps coming from $\mathcal{F}$ over X. This defines a functor $f_* : \mathrm{PSh}(X) \to \mathrm{PSh}(Y)$, and one may easily verify that this preserves the full subcategories of sheaves on X and Y.

Example 2.7. For the terminal one point space $\bullet$, we may recognise the push-forward of $\mathcal{F}$ along the terminal map $t : X \to \bullet$ as a sheaf over $\bullet$, the data of which is just an object of C. This object is given by $t_*\mathcal{F}(\bullet) = \mathcal{F}(t^{-1}(\bullet)) = \mathcal{F}(X)$, which are just the global sections of $\mathcal{F}$.

Definition 2.8 (Inverse image)**.** For a presheaf $\mathcal{F}$ on Y, we define the inverse image presheaf $f^{-1}\mathcal{F}$ to have sections $f^{-1}\mathcal{F}(U) = \operatorname{colim}_{f(U)\subset V} \mathcal{F}(V)$, with restriction maps induced from this colimit. This assignment on objects, along with the induced maps on morphisms yields a functor $f^{-1} : \mathrm{PSh}(Y) \to \mathrm{PSh}(X)$.

Remark 2.9. The inverse image of a sheaf need not be a sheaf in general. Two points mapping to one gives a counterexample, for any nonempty sheaf on the one point space.

We may describe two fundamental operations on sheaves in terms of the pullback functor, associated with natural maps into X. The first of these is the pullback associated to an open set inclusion:

Definition 2.10 (Restriction of a sheaf)**.** Given an open inclusion $U \xrightarrow{i} X$, we define the restriction of $\mathcal{F}$ along U as $\mathcal{F}|_U := i^{-1}\mathcal{F}$. Concretely, we have $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for $V \subset U$.

The second is the pullback associated to the inclusion of a point $x \xrightarrow{i_x} X$.

Definition 2.11 (Stalk of a sheaf)**.** Given a C sheaf $\mathcal{F}$ on a topological space X, the stalk of $\mathcal{F}$ at x is the C object $\mathcal{F}_x := i_x^{-1}\mathcal{F}(\{x\}) = \operatorname{colim}_{x\in U} \mathcal{F}(U)$.

The stalk of $\mathcal{F}$ at a point x can be thought of as detecting the behaviour of sections of $\mathcal{F}$ in an "arbitrarily small" neighbourhood of the point x. Given a map $\mathcal{F} \to \mathcal{G}$ of sheaves over X, we get a naturally induced map on stalks $\mathcal{F}_x \to \mathcal{G}_x$.

Theorem 2.12. *For a continuous map $f : X \to Y$ of topological spaces, the functor f^{-1} is a left adjoint of f_*.*

$$\mathrm{PSh}(Y) \underset{f_*}{\overset{f^{-1}}{\rightleftarrows}} \mathrm{PSh}(X)$$

Proof. We have natural transformations $\varepsilon_{\mathcal{F}} : f^{-1}f_*\mathcal{F} \to \mathcal{F}$ given by the canonical map induced from the the compatible family of restriction maps

$$\operatorname{colim}_{f(U)\subset V} f_*\mathcal{F}(V) = \operatorname{colim}_{f(U)\subset V} \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U).$$

Since $f(f^{-1}(U)) \subset U$ for all U, we have a natural transformation $\eta_{\mathcal{G}} : \mathcal{G} \to f_* f^{-1}\mathcal{G}$ given by the canonical map

$$\mathcal{G}(U) \to \operatorname{colim}_{f(f^{-1}(U))\subset K} \mathcal{G}(K).$$

These are the counit and unit of the adjunction. So we need to verify that $f_*(\varepsilon_{\mathcal{G}}) \circ \eta_{f_*\mathcal{G}} = \mathrm{id}_{f_*\mathcal{G}}$ and $\varepsilon_{f^{-1}\mathcal{F}} \circ f^{-1}(\eta_{\mathcal{F}}) = \mathrm{id}_{f^{-1}\mathcal{F}}$.

This is to say, the following composites are identities.

$$\begin{array}{ccc} \mathcal{G}(f^{-1}(U)) & \longrightarrow & \operatorname{colim}_{f(f^{-1}(U))\subset K} \mathcal{G}(f^{-1}(K)) \\ & \searrow & \downarrow \\ & & \mathcal{G}(f^{-1}(U)) \end{array}$$

$$\begin{array}{ccc} \operatorname{colim}_{f(U)\subset V} \mathcal{F}(V) & \longrightarrow & \operatorname{colim}_{f(U)\subset V} \operatorname{colim}_{f(f^{-1}(V))\subset K} \mathcal{F}(K) \\ & \searrow & \downarrow \\ & & \operatorname{colim}_{f(U)\subset V} \mathcal{F}(V) \end{array}$$

For the first of these, note that $f^{-1}(U) = f^{-1}(f(f^{-1}(U))) \subset f^{-1}(K)$. The compatible family of maps given by restriction from $f^{-1}(K)$ induces the vertical map, so the first dashed arrow is the identity, being the restriction map from $\mathcal{G}(f^{-1}(U))$ to $\mathcal{G}(f^{-1}(U))$.

The second map is similar, since $f(U) \subset V$, and $f(f^{-1}(V)) \subset K$, so we have

$$f(U) = f(f^{-1}(f(U))) \subset f(f^{-1}(V)) \subset K.$$

The canonical maps from $\mathcal{F}(V)$ to $\operatorname{colim}_{f(U)\subset V} \mathcal{F}(V)$ factor through this double colimit, and the unique induced map is therefore the identity.

□

Now that we know what sheaves are, we should see how to construct them. Our first method of constructing sheaves is to canonically build a sheaf out of a presheaf.

Construction 2.13 (Sheafification)**.** Let $\mathcal{F}$ be a C presheaf on X, where C is a concrete category. If $s \in \mathcal{F}(U)$, and $x \in U$ we denote the image of s in the stalk at x by $s|_x$. We define the sheafification $\mathrm{Sh}(\mathcal{F})$ of $\mathcal{F}$ to be the sheaf on X with sections over U given by

$$\operatorname{Sh}\mathcal{F}(U) = \left\{ (s_x)_{x\in U} \in \prod_{x\in X} \mathcal{F}_x \;\middle|\; \begin{array}{c} \forall x \in U, \text{ there exists } U \supset V \ni x \text{ and } s \in \mathcal{F}(V), \\ \text{such that } s|_y = s_y \text{ for all } y \in V. \end{array} \right\}.$$

With restriction maps the projections $\prod_{x \in U} \mathcal{F}_x \to \prod_{x \in V} \mathcal{F}_x$. It is straightforward to check that this gives a sheaf on X, and by taking the product of the passage to stalk maps $s \to \prod_{x \in U} s|_x$, we obtain a map $\mathcal{F} \xrightarrow{sh} \operatorname{Sh} \mathcal{F}$.

We may interpret this map $\mathcal{F} \xrightarrow{sh} \operatorname{Sh} \mathcal{F}$ as a two step process of turning a presheaf into a sheaf. First, we force the sections of $\operatorname{Sh} \mathcal{F}$ to be determined by their values on stalks, by viewing a section as nothing more than its stalkwise values. Since this need not hold for sections of an arbitrary presheaf, but all sheaves do have this property, we can liken this step to throwing out the "bad" sections of $\mathcal{F}$. Then, we require compatible families of local data to glue, so we just include all the families of local data from $\mathcal{F}$ that are compatibly defined on an open cover $\{U_i\}_{i \in I}$ of U.

Theorem 2.14. *The inclusion ι of the full subcategory $\operatorname{Sh}(X)$ into $\operatorname{PSh}(X)$ admits a left adjoint, the sheafification functor.*

$$\operatorname{PSh}(X) \underset{\iota}{\overset{\operatorname{Sh}}{\rightleftarrows}} \operatorname{Sh}(X)$$

Proof. Let $\mathcal{F}$ be a presheaf, and $\mathcal{G}$ a sheaf on X. We need to check that for any morphism $\mathcal{F} \xrightarrow{\phi} \mathcal{G}$ that there exists a unique morphism $\operatorname{Sh} \mathcal{F} \xrightarrow{\tilde{\phi}} \mathcal{G}$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\phi} & \mathcal{G} \\ {\scriptstyle sh}\downarrow & \nearrow_{\tilde{\phi}} & \\ \operatorname{Sh} \mathcal{F} & & \end{array}$$

To define this map $\tilde{\phi}$, express any section $s \in \operatorname{Sh} \mathcal{F}(U)$ as a compatible family of sections $s_i \in \mathcal{F}(U_i)$ on an open cover of $\{U_i\}_{i \in I}$ of U, and map s to the unique element of $\mathcal{G}(U)$ corresponding to the compatible family $\phi(s_i) \in \mathcal{G}(U_i)$. This is independent of the s_i and U_i used to represent s, since it is stable under refinement, as $\mathcal{G}$ is a sheaf. □

Presheaves are a much more robust objects than sheaves, any functor from C to D results in a method of producing D valued presheaves from C valued presheaves. By sheafifying, we may extend this to sheaves, giving us a way of constructing new sheaves from old.

As an instance of this, we may now extend our inverse image functor to the categories of sheaves.

Corollary 2.15. *For a continuous map of spaces $f : X \to Y$, the pushforward functor of sheaves admits a left adjoint $\operatorname{Sh} \circ f^{-1}$.*

$$\mathrm{Sh}(Y) \underset{f_*}{\overset{\mathrm{Sh}\circ f^{-1}}{\rightleftarrows}} \mathrm{Sh}(X)$$

Proof. Adjunctions are closed under composition, so this follows from Theorems 2.12 and 2.14. □

From here, we will let f^{-1} denote the adjoint of f_* on the sheaf categories, letting the sheafification be implied. We will be working primarily with categories of sheaves, so this distinction will hopefully not cause confusion. All of our data categories C will now be taken to be concrete, so we will speak of sections of $\mathcal{F}$ over U as elements of sets.

Sheafification is a global process, and thus it is reassuring that the "local information" of a presheaf is preserved.

Proposition 2.16. *The canonical map $\mathcal{F} \xrightarrow{sh} \mathrm{Sh}(\mathcal{F})$ induces an isomorphism $\mathcal{F}_x \cong \mathrm{Sh}(\mathcal{F})_x$ on stalks.*

Proof. Sheafification is a left adjoint, so preserves the colimit used to define $\mathcal{F}_x$. □

We will repeatedly use the following corollaries of the sheafification construction.

Corollary 2.17. *A section of $\mathcal{F}$ over U is determined by its value on stalks $\mathcal{F}_x$ for $x \in U$.*

Proof. The sections of $\mathrm{Sh}(\mathcal{F})$ over U are naturally a subset of $\prod_{p\in U} \mathcal{F}_p$, so for a sheaf, since $\mathrm{Sh}(\mathcal{F}) \cong \mathcal{F}$, we have that the canonical map $\mathcal{F}(U) \to \prod_{p\in U} \mathcal{F}_p$ is an injection. □

Corollary 2.18. *If $\mathcal{F}$ is a presheaf on X, such that $\mathcal{F}|_U$ is a sheaf on U open in X, then $(\mathrm{Sh}\,\mathcal{F})|_U \cong \mathcal{F}|_U$.*

Proof. The sheafification of $\mathcal{F}$ on U depends only on the stalks at points in U, with a condition that can be checked locally in U. Thus, the sheafification on U is dependent only on $\mathcal{F}|_U$, giving the claim. □

This corollary leads into our other construction of sheaves. Given a family of sheaves $\mathcal{F}_i$ defined on an open cover $\{U_i\}_{i\in I}$ of X, we may "glue" them to define a sheaf on X.

Proposition 2.19. *Let $\mathcal{F}_i$ be sheaves on $\mathcal{U} = \{U_i\}_{i \in I}$ an open cover of X, with isomorphisms*

$$\gamma_{i,j} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$$

as sheaves on $U_i \cap U_j$, such that $\gamma_{j,k} \circ \gamma_{i,j} = \gamma_{i,k}$ on $U_i \cap U_j \cap U_k$. Then there exists a unique sheaf $\mathcal{F}$ on X such that $\mathcal{F}|_{U_i} \cong \mathcal{F}_i$, unique up to isomorphism.

Proof. We define $\mathcal{F}(U)$ to be the equaliser

$$\mathcal{F}(U) \to \prod_i \mathcal{F}_i(U \cap U_i) \rightrightarrows \prod_{k,j} \mathcal{F}_k(U \cap U_k \cap U_j).$$

Here the (k, j) component of the first of the double arrows is given by is projecting onto the kth coordinate, then restricting onto $U \cap U_k \cap U_j$. The (k, j) component of the second map is given by projecting onto the jth component, restricting onto $U \cap U_j \cap U_k$, then applying the isomorphism $\gamma_{j,k}$.

To check that this is a sheaf with the desired properties, let $\{V_m\}_{m \in J}$ be an open cover of V in X, and consider the following diagram:

$$\begin{array}{ccccc}
\mathcal{F}(V) & \longrightarrow & \prod_i \mathcal{F}_i(V \cap U_i) & \rightrightarrows & \prod_{k,j} \mathcal{F}_k(V \cap U_k \cap U_j) \\
\downarrow & & \downarrow & & \downarrow \\
\prod_n \mathcal{F}(V_n) & \longrightarrow & \prod_n \prod_i \mathcal{F}_i(V_n \cap U_i) & \rightrightarrows & \prod_n \prod_{k,j} \mathcal{F}_k(V_n \cap U_k \cap U_j) \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
\prod_{n,m} \mathcal{F}(V_n \cap V_m) & \longrightarrow & \prod_{n,m} \prod_i \mathcal{F}_i(V_n \cap V_m \cap U_i) & \rightrightarrows & \prod_{n,m} \prod_{k,j} \mathcal{F}_k(V_n \cap V_n \cap U_k \cap U_j)
\end{array}$$

To show that this $\mathcal{F}$ is a sheaf, we need the far left vertical fork to be an equaliser diagram. For this, observe that every other fork in the diagram is an equaliser, either by definition, or since it is a product of equalisers. From this, it is an enjoyable diagram chase to show that the left fork is also an equaliser, which we leave to the reader.

The reader may note that we haven't yet used the compatibility conditions on our $\gamma_{i,j}$. We will need them now to check that the restriction of $\mathcal{F}$ to U_ℓ is isomorphic to $\mathcal{F}_\ell$.

For this, we have the following natural diagram for any $V \subset U_\ell$:

$$\begin{array}{ccccc}
\mathcal{F}_\ell(V) & \longrightarrow & \prod_i \mathcal{F}_\ell(V \cap U_i) & \rightrightarrows & \prod_{j,k} \mathcal{F}_\ell(V \cap U_j \cap U_k) \\
\downarrow & & \downarrow \prod_i \gamma_{\ell,i} & & \downarrow \prod_k \gamma_{\ell,k} \\
\mathcal{F}(V) & \longrightarrow & \prod_i \mathcal{F}_i(V \cap U_i) & \rightrightarrows & \prod_{j,k} \mathcal{F}_k(V \cap U_j \cap U_k)
\end{array}$$

Our compatibility condition on the $\gamma_{i,j}$ ensures that this diagram commutes, so since the vertical maps are natural isomorphisms, we see that $\mathcal{F}_\ell \cong \mathcal{F}|_{U_\ell}$.

□

To express this construction more categorically, fix an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of a space X.

Definition 2.20. The category of sheaves on $\mathcal{U}$ with descent data, denoted $\mathrm{Sh}_{\mathcal{U}}(X)$ has objects $(\mathcal{F}_i, \gamma_{i,j})_{i,j \in I}$, where $\mathcal{F}_i$ is a sheaf on U_i, with isomorphisms

$$\gamma_{i,j} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$$

as sheaves on $U_i \cap U_j$, such that $\gamma_{j,k} \circ \gamma_{i,j} = \gamma_{i,k}$ on $U_i \cap U_j \cap U_k$. A morphism in this category

$$(\mathcal{F}_i, \gamma_{i,j}) \to (\mathcal{G}_i, \tau_{i,j})$$

is a family of morphisms $\phi_i : \mathcal{F}_i \to \mathcal{G}_i$ such that that following diagram commutes for all i.

$$\begin{array}{ccc} \mathcal{F}_i|_{U_i \cap U_j} & \xrightarrow{\phi_i|_{U_i \cap U_j}} & \mathcal{G}_i|_{U_i \cap U_j} \\ \downarrow{\scriptstyle \gamma_{i,j}} & & \downarrow{\scriptstyle \tau_{i,j}} \\ \mathcal{F}_j|_{U_i \cap U_j} & \xrightarrow{\phi_j|_{U_i \cap U_j}} & \mathcal{G}_j|_{U_i \cap U_j} \end{array}$$

We have a natural functor from $\mathrm{Sh}(X)$ to $\mathrm{Sh}_{\mathcal{U}}(X)$ given by taking $\mathcal{F}$ to its restrictions $\mathcal{F}|_{U_i}$, along with the canonical maps induced by the equalities

$$(\mathcal{F}|_{U_i})|_{U_j}(V) = (\mathcal{F}|_{U_j})|_{U_i}(V).$$

Without much difficulty, we may upgrade Proposition 2.19 into the following equivalence of categories.

Theorem 2.21. *The natural functor* $\mathrm{Sh}(X) \to \mathrm{Sh}_{U_i}(X)$ *is an equivalence of categories.*

We will only use this theorem explicitly in chapter 4, but implicitly this viewpoint on sheaves will be present throughout this thesis.

2.2 Affine schemes

We want to think of spaces and their "functions" together, so our algebra-geometric objects will be pairs, (X, O_X) consisting of a space X, and a sheaf of rings on that space, which we will think of as the functions on X.

First, we need to recall the definition of a local ring from commutative algebra.

Definition 2.22 (Local ring)**.** A ring R is local if it has a unique maximal ideal $\mathfrak{m}$, and a morphism of local rings $\phi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a local morphism if $\phi(\mathfrak{m}) \subset \mathfrak{n}$.

Definition 2.23 (Locally ringed space)**.** A locally ringed space is a pair (X, O_X) of a topological space X, with a sheaf of rings O_X on X, such that for each point x of X, the stalk $O_{X,x}$ is a local ring. The sheaf of rings O_X is called the structure sheaf of X.

Example 2.24. The sheaf of continous functions endows (X, C_X) with the structure of a locally ringed space. We already saw that this is a sheaf of rings, so it remains to check that the stalks are local rings. The stalk at a point $x \in X$ consists of functions f defined in a neighbourhood of x, where we identify f and g if they agree on a potentially smaller neighbourhood of x. The functions that vanish at x are the kernel of the evaluation at x map $(C_X)_x \xrightarrow{ev_x} \mathbb{R}$, and thus form a maximal ideal of $(C_X)_x$. To check that $(C_X)_x$ is a local ring, note that if a function f doesn't vanish at x, then it doesn't vanish in a neighbourhood of x, so it is invertible in a neighbourhood of x.

Given that any continous map $f : X \to Y$ induces a pullback on continous functions, we can interpret this situation to give the definition of a morphism of locally ringed spaces.

Definition 2.25 (Morphism of locally ringed spaces)**.** A morphism of locally ringed spaces $(f, f^\times) : (X, O_X) \to (Y, O_Y)$ consists of the data of a continuous map $f : X \to Y$, and a morphism of sheaves of rings $f^\times : O_Y \to f_* O_X$, such that the induced maps $O_{Y,f(P)} \to (f_* O_X)_{f(P)} \to O_{X,P}$ on stalks are local ring homomorphisms.

When the map on structure sheaves is clear from context, by abuse of notation, we refer to morphisms of locally ringed spaces by their map of underlying spaces.

Given an open set $U \subset X$, we may restrict the structure sheaf of X to endow U with the structure of a locally ringed space.

Proposition 2.26. *For an open subset U of X, the pair $(U, O_X|_U)$ is a locally ringed space.*

Proof. The sheaf $O_X|_U$ is a sheaf of rings, and since we may compute stalks locally in U, we see that $(U, O_X|_U)$ is a locally ringed space. □

The notion of a locally ringed space gives a generalisation of how functions defined on a space behave. We have rings of "functions", defined on open subsets of a space, and those rings have "values" at points by interpreting the canonical map $O_{X,P} \to O_{X,P}/\mathfrak{m}$ as the "evaluation at x" map.

We aim now to construct, for any ring R, a locally ringed space $\mathrm{Spec}(R)$ for which R is "the functions on $\mathrm{Spec}(R)$". Let $R_\mathfrak{p}$ denote the localisation of R at a prime ideal $\mathfrak{p}$, for a refresher on localisation, see [AM69]. We would like to interpret the local ring $R_\mathfrak{p}$ as the germs of functions at a point, with $R_\mathfrak{p} \to R_\mathfrak{p}/\mathfrak{p}$ the "evaluation at $\mathfrak{p}$" map, so the points of our space ought to be prime ideals of R.

To topologise the set of prime ideals, we need to specify which sets are open and closed. Since our "evaluation" maps $R_\mathfrak{p} \to R_\mathfrak{p}/\mathfrak{p}$ land in different fields, we cannot rely on properties of any specific field. However, we may note that while the "values" of f at the different primes may not be comparable, the notion of "vanishing at $\mathfrak{p}$" is well defined. We may take f vanishing at $\mathfrak{p}$ to mean the coset $\bar{f}$ in $R_\mathfrak{p}/\mathfrak{p}$ is zero, or more simply, that $f \in \mathfrak{p}$.

So if we want these vanishing sets to be closed, we should have the sets

$$Z_f := \{\mathfrak{p} \in \mathrm{Spec}(R) | f \in \mathfrak{p}\}$$

closed in $\mathrm{Spec}(R)$.

Definition 2.27. The topological space $\mathrm{Spec}(R)$ is the set of prime ideals of R, with the topology generated by the closed sets Z_f.

This topology is called the Zariski topology, and we have the following explicit description of it.

Lemma 2.28. *The Zariski topology on* $\mathrm{Spec}(R)$ *has closed sets*

$$Z_I = \{\mathfrak{p} \in \mathrm{Spec}(R) | I \subset \mathfrak{p}\}$$

where I is an arbitrary ideal of R.

Proof. First, note that $Z_I = \bigcap_{f \in I} Z_f$, since a prime ideal $\mathfrak{p}$ contains I if and only if it contains all the elements of I, that is, $\mathfrak{p} \in Z_f$. So it suffices to check that these closed sets form a topology. This follows by verifying these simple properties of Z_I.

i) $Z_I \cup Z_J = Z_{I \cap J}$

ii) $\bigcap_j Z_{I_j} = Z_{\sum_j I_j}$

iii) $Z_{(0)} = \mathrm{Spec}(R)$

iv) $Z_{(1)} = \emptyset$

□

Lemma 2.28 shows that $D_f := \mathrm{Spec}(R) \setminus Z_f$ are a basis for the topology on $\mathrm{Spec}(R)$. We call open subsets of this form distinguished open subsets of $\mathrm{Spec}(R)$. Similarly, we define the open sets D_I by

$$D_I := \mathrm{Spec}(R) \setminus Z_I.$$

So for this space, we would like to interpret R as the global sections of a sheaf of rings O_R on $\mathrm{Spec}(R)$. To determine this structure sheaf for other open subsets, we should observe the fact that in topology, a real valued continuous function f on an open set U is invertible if it doesn't vanish on any point in U. This is a perfectly well behaved algebraic notion too, so as a first attempt, we may construct the "functions on D_I" by taking R, and inverting those $f \in R$ which have Z_f contained in Z_I.

Algebraically, this is setting

$$\tilde{O}_R(D_I) := R[f_i^{-1}]$$

for all f_i such that $Z_f \in Z_I$.

Interpreting the natural maps $\tilde{O}_R(D_I) \to \tilde{O}_R(D_J)$ as restriction maps, we arrive at a presheaf $\tilde{O}_R$ on $\mathrm{Spec}(R)$. Unfortunately, this is not a sheaf in general, so we then sheafify $\tilde{O}_R$ to obtain a sheaf of rings O_R.

Proposition 2.29. *The pair* $(\mathrm{Spec}(R), O_R)$ *is a locally ringed space.*

Proof. We have a topological space and a sheaf of rings, so we just need to check that the stalks of O_R are local rings. Since sheafification preserves stalks, it suffices to check that the stalks of $\tilde{O}_R$ are local rings. We would like the stalk at $\mathfrak{p}$ to be the local ring $R_{\mathfrak{p}}$. For this, we have

$$(O_R)_{\mathfrak{p}} := \mathrm{colim}_{\mathfrak{p} \in D_I} \tilde{O}_R(D_I) = \mathrm{colim}_{\mathfrak{p} \in D_I} \mathrm{colim}_{Z_f \subset Z_I} R[f^{-1}]$$

We may rewrite this last colimit as the colimit of $R[f^{-1}]$ over all f such that there exists an ideal I not contained in $\mathfrak{p}$, such that $Z_f \subset Z_I$. We claim that this condition is satisfied if and only if f is not contained in $\mathfrak{p}$. The if direction is clear, take $I = (f)$. For the converse, if $Z_f \subset Z_I$, then since $\sqrt{(f)} = \bigcap_{(f)\subset\mathfrak{p}} \mathfrak{p}$, we have $I \subset \sqrt{(f)}$, so if I is not contained in a given prime ideal $\mathfrak{q}$, then neither is f.

Thus, this last colimit is exactly $R_\mathfrak{p}$. □

Now that we have our locally ringed space $(\operatorname{Spec}(R), O_R)$, it remains to prove our original desired property, that the global sections of O_R are precisely R.

Theorem 2.30. *The global sections of* $(\operatorname{Spec}(R), O_R)$ *can be canonically identified with* R.

Proof. Now from 2.29 we know the stalks of O_R, and we have that D_f are a basis of our topology, we may describe the sections of O_R explicitly as

$$O_R(U) = \left\{ (x_\mathfrak{p} \in R_\mathfrak{p})_{\mathfrak{p}\in U} \;\middle|\; \begin{array}{c} \text{for all } \mathfrak{p} \in U, \text{ there exists } f \in R, g \in R[1/f] \\ \text{with } \mathfrak{p} \in D_f, \text{ such that } x_\mathfrak{q} = g|_\mathfrak{q} \text{ for all } \mathfrak{q} \in D_f. \end{array} \right\}.$$

That is, the sections over U are families of elements in each $R_\mathfrak{p}$, compatible in the sense that they look like restrictions of functions defined on our base of distinguished subsets D_f.

Let's now check that the only global sections of O_R are the "obvious" ones, coming from elements of $\tilde{O}_R(\operatorname{Spec}(R)) \cong R$.

First, let's note that R injects into the global sections of O_R. If $f \in R$ is in the kernel of this map, then f is zero in $R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Thus we have, for each prime ideal $\mathfrak{p}_i$, an element s_i, with $s_i \notin \mathfrak{p}_i$ and $s_i f = 0$. The ideal generated by these s_i is not contained in any prime ideal, so is the unit ideal, so we have

$$\sum_{i=1}^{n} a_i s_i = 1$$

for some $a_i \in R$. Multiplying both sides by f yields that $f = 0$.

For surjectivity, if we have a global section s, then from our description of O_R, we have an open cover $\{D_{f_i}\}_{i\in I}$, and elements $g_i \in R[1/f_i]$ such that

$$s = g_i \text{ on } D_{f_i}, \text{ with } g_i = g_j \text{ on } D_{f_i f_j}$$

. Since the D_{f_i} are an open cover of $\operatorname{Spec}(R)$, the ideal generated by the f_i is the unit ideal, so we have a finite R-linear combination of them that sum to $1 \in R$. Thus, we may assume that our index set I is finite.

Each g_i can be expressed as $\frac{a_i}{f_i^{n_i}}$, and by changing our representative of g_i, we may assume all $n_i = N = \max_{i\in I}\{n_i\}$, so we may express g_i as

$$g_i = \frac{b_i}{f_i^N}.$$

Our compatibility condition gives the existence of $K_{ij} \in \mathbb{N}$ with

$$(f_i f_j)^{K_{ij}}(b_i f_j^N - b_j f_i^N) = 0.$$

Now set $K := \max_{i,j\in I}(K_{ij})$, and note that since our f_i generate the unit ideal, there exist $c_i \in R$ such that

$$\sum_{i=1}^{n} c_i f_i^{N+K} = 1.$$

Then set $d := \sum_{i=1}^{n} c_i f_i^K b_i$. By construction, we have

$$f_j^{N+K} d = f_j^{N+K} \sum_{i=1}^{n} c_i f_i^K b_i = \sum_{i=1}^{n} c_i f_i^K f_j^{N+K} b_i = \sum_{i=1}^{n} c_i f_i^{N+K} f_j^K b_j = f_j^K b_j.$$

So for each $j \in I$, on D_j we have

$$g_j = \frac{b_j}{f_j^N} = d.$$

So $d = s$, and the global sections are precisely R.

□

Applying this argument to each D_f gives that

$$O_R(D_f) = R[1/f].$$

Since these form a base of the topology, we see that the sections on an arbitrary U are just compatible families of elements in $R[1/f_i]$ as for $\{D_{f_i}\}_{i\in I}$ a cover of U.

Remark 2.31. We could have given the Zariski topology and the explicit description of the structure sheaf directly to construct $(\mathrm{Spec}(R), O_R)$, but we opted for the more roundabout approach for its intuitive appeal and perceived pedagogical value.

Given a ring homomorphism $R \xrightarrow{\phi^\times} S$, the preimage of a prime ideal is prime, so we have an induced map $\mathrm{Spec}(S) \xrightarrow{\phi} \mathrm{Spec}(R)$ of topological spaces. This is easily seen to be continuous since $\phi^{-1}(Z_f) = Z_{\phi^\times(f)}$. By localising, we may upgrade the map $R \xrightarrow{\phi^\times} S$ to a map of structure sheaves $O_R \to \phi_* O_S$.

The construction Spec is therefore functorial, and yields a functor

$$\mathrm{Spec} : \mathrm{Ring}^{op} \to \text{Locally ringed spaces}.$$

From this perspective, we see that the canonical morphism $R \to R[\frac{1}{f}]$ inverting an element $f \in R$ corresponds to restricting $(\mathrm{Spec}(R), O_R)$ to the sub-locally ringed space $(D_f, O_R|_{D_f})$.

Regarding this functor, we have the following result, a proof of which can be found as proposition 2.2.3 of [Har77].

Theorem 2.32. *The functor* Spec *is fully faithful, so yields an equivalence of categories between Ring^{op} and its essential image in the category of locally ringed spaces.*

So we can intepret the category Ring^{op} as naturally living inside the category of locally ringed spaces.

Definition 2.33 (Affine schemes)**.** An affine scheme is a locally ringed space isomorphic to $(\mathrm{Spec}(R), O_R)$ for some ring R.

Via this equivalence, we may attach geometric meaning to properties and phenomena in the category of rings. The most important of these is the definition of a closed subscheme, the dual notion to a surjective map of rings.

Definition 2.34 (Closed subscheme of an affine scheme)**.** A closed subscheme of $\mathrm{Spec}(R)$ is a sub locally ringed space $Z \subset \mathrm{Spec}(R)$ corresponding to Spec of a surjection of the form $R \to R/I$. A closed immersion $X \to \mathrm{Spec}(R)$ is a morphism of locally ringed spaces that factors as an isomorphism, followed by the inclusion of a closed subscheme.

From Theorem 2.32 we have a order reversing bijection from ideals of R to closed subschemes of $\mathrm{Spec}(R)$.

Note that closed subschemes of affine schemes are themselves affine schemes, since they are isomorphic to $\mathrm{Spec}(R/I)$. On the underlying spaces, a closed subscheme inclusion is given by a closed subset inclusion.

Remark 2.35. Closed subschemes are very different from topological closed set inclusions however. For instance, a nontrivial field extension induces a homeomorphism on the underlying topological spaces, but is not a closed immersion. The closed subscheme inclusions $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ and $\mathrm{Spec}(R/I^2) \to \mathrm{Spec}(R)$ are potentially distinct also, but give the same underlying closed subset inclusion.

2.3 Schemes

Our geometric objects of interest will be modelled on spaces of the form $\mathrm{Spec}(R)$.

Definition 2.36 (Schemes)**.** A scheme is a locally ringed space that is locally isomorphic to an affine scheme, that is, admits an open cover $\{U_i\}_{i\in I}$ such that $(U_i, O_{X|_{U_i}})$ are affine schemes.

For a scheme (X, O_X), we denote the stalk at a point x by $O_{X,x}$.

Example 2.37. Any affine scheme $(\mathrm{Spec}(R), O_R)$ is a scheme, taking the open cover to consist of $\mathrm{Spec}(R)$ itself.

Example 2.38. Less trivially, for any affine scheme (X, O_X) the sub-locally ringed space $(U, O_X|_U)$ is a scheme. This follows from 2.30, since each $(D_f, O_X|_{D_f}) \cong (\mathrm{Spec}(R[\frac{1}{f}]), O_{R[\frac{1}{f}]})$. Schemes of this type need not be affine when U is D_I for a nonprincipal ideal I of R.

We call a sub-locally ringed space of the form $(U, O_X|_U)$ an open subscheme of (X, O_X).

Definition 2.39 (Closed immersion of schemes)**.** A closed immersion of schemes is a morphism $i : Z \to X$ such that each $i : Z|_{f^{-1}(U)} \to U$ is a closed immersion for U affine in X. Similarly, a closed subscheme of X is closed immersion which is the inclusion of a closed subscheme on each affine U.

Since schemes are locally affine, and affine schemes are dual to rings, many properties of rings translate to this global setting. For us, all of our schemes will have these properties, to avoid some complications and pathologies.

Definition 2.40 (Reduced schemes)**.** A scheme X is reduced if the rings $O_X(U)$ all have trivial nilradical.

Definition 2.41 (Irreducible schemes)**.** A scheme X is irreducible if its underlying topological space is irreducible, that is, cannot be expressed as the union of two proper closed sets.

Definition 2.42 (Integral schemes)**.** A scheme X is integral if it is both reduced and irreducible.

Definition 2.43 (Noetherian schemes)**.** A scheme X is Noetherian if every descending chain of closed subsets stabilises.

Note that if a scheme X has these properties, then any open affine U in X is dense, and also has these properties. Furthermore, in the affine case, $\mathrm{Spec}(R)$ having these properties is equivalent to R being a Noetherian integral domain.

The next ring theoretic construction we wish to globalise is the field of fractions associated to an integral domain.

Proposition 2.44. *For any integral scheme X, there is a unique point such that its closure is X, and this point is the intersection of all open sets of X.*

Proof. Let U be an open affine subscheme of X, and g_U the point of U corresponding to the zero ideal in $O_U(U)$. Since the closure of g_U contains U, and X is irreducible, we have that the closure of g_U is all of X.

For uniqueness, first note that there is a unique point with this property in any affine subscheme Spec(A), which corresponds to the unique minimal ideal of A

$$(0) = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p}.$$

Now, given another point x, with the closure of x being X, x cannot be contained in $X \setminus U$, so $x \in U$, giving $x = g_U$.

If y is any point contained in every open subset of X, then its closure is a closed set which has nonempty intersection with every nonempty open subset of X, so must be all of X. □

Definition 2.45. The generic point of an integral scheme X is defined to be the unique point x_{gen} such that the closure of x_{gen} is X.

Definition 2.46 (Field of fractions)**.** The field of fractions $K(X)$ of an integral scheme X is the stalk of O_X at x_{gen}.

As the name would suggest, the field of fractions $K(X)$ is a field. The inclusion of an open affine $U \to X$ induces an isomorphism on the stalks at points in U, so

$$O_{X x_{gen}} \cong \mathrm{Quot}(O_U(U)).$$

2.4 Dedekind domains

Our schemes will be built out of particularly well behaved rings, Dedekind domains.

Definition 2.47 (Dedekind domain)**.** A Dedekind domain is an integral domain A which is Noetherian, has Krull dimension one, and integrally closed in its field of fractions.

We will be using Dedekind domains extensively, and we refer the reader to chapter 9 of [AM69] for a proof of the following equivalences.

Theorem 2.48. *The following are equivalent for a domain A:*

i) A is integrally closed in its field of fractions, Noetherian, and has Krull dimension 1.

ii) Every nonzero ideal I in A can be expressed uniquely up to reordering of factors as a finite product of powers of prime ideals, $I = \prod_{i=1}^{n} P^{e_i}$.

iii) A is Noetherian, and the localisation $A_{\mathfrak{p}}$ is a principal ideal domain for all nonzero prime ideals $\mathfrak{p}$.

To understand the local rings $A_{\mathfrak{p}}$ for A a Dedekind domain, we will use discrete valuations.

Definition 2.49 (Discrete valuation on a ring)**.** A discrete valuation v on a ring R is a map $v : R \to \mathbb{Z} \cup \{-\infty\}$ such that:

i) $v(ab) = v(a) + v(b)$

ii) $v(a + b) \geq \min(v(a), v(b))$

iii) $v(x) = -\infty$ if and only if $x = 0$

Since we will not being dealing with valuations that are not discrete in this thesis, we will often refer to discrete valuations simply as valuations. Associated to any discrete valuation v, we have an associated ring.

Definition 2.50. The subring O_v associated to a discrete valuation v on R is the subring defined by

$$O_v = \{x \in R | v(x) \geq 0\}.$$

Within this subring, we have a distinguished prime ideal $\mathfrak{m}$ given by

$$\mathfrak{m} = \{x \in R | v(x) > 0\}.$$

We say that two discrete valuations v, w are equivalent, denoted $v \sim w$, if $w(x) = q \cdot v(x)$ for some $q \in \mathbb{Q}$, for all $x \in R$. The connection to Dedekind domains is given by the following proposition.

Proposition 2.51. *A local principal ideal domain $(A, \mathfrak{m})$ has a canonical valuation v defined by $(x) = \mathfrak{m}^{v_{\mathfrak{m}}(x)}$. If K is a field with a nonzero valuation v, then O_v is a local principal ideal domain, and the the associated valuation is equivalent to v.*

Proof. The fact that $v_{\mathfrak{m}}$ is a valuation is easily verified from the properties of Dedekind domains given by Theorem 2.48. For the second claim, pick an element π of O_v such that $v(\pi)$ is positive and minimal, with $v(\pi) = k$. By property i) of definition 2.49, and minimality, we see that any element $x \in O_v$ has $v(x)$ divisible by k, so we may normalise v to have $v(\pi) = 1$. So now if x has $v(x) = n$, then $v(x\pi^{-n}) = 0$, so $x\pi^{-n}$ is a unit in O_v, so every element $x \in O_v$ is of the form $\pi^n u$ for u a unit in O_v, so O_v is a local principal ideal domain. □

Definition 2.52 (Discrete valuation ring)**.** We define a discrete valuation ring to be a local principal ideal domain. We see from Proposition 2.51 that any discrete valuation ring comes with a canonical valuation, which we denote by v.

We have seen that elements of valuation equal to 1 play a special role in the theory of discrete valuation rings, in that they generate the maximal ideal $\mathfrak{m}$.

Definition 2.53 (Uniformising parameter)**.** For a discrete valuation ring R, a uniformising parameter, or uniformiser of R is an element π of R with $v(\pi) = 1$. Equivalently, a generator of the maximal ideal of R.

Throughout this thesis, we will use π to denote a uniformising parameter of a discrete valuation ring.

Finally, observe that if R is a discrete valuation ring inside its field of fractions K, then every proper nonzero R submodule of R is free of rank one, generated by π^n for some $n \in \mathbb{N}$. For $\mathfrak{p}$ the maximal ideal of R, we will write $\mathfrak{p}^n$ to mean the submodule of K generated by π^n for $n \in \mathbb{Z}$. Note we allow n to be negative, so this need not be an ideal of R.

2.5 Dedekind schemes

We need one more general property of schemes before defining our schemes of interest, that of seperating points.

Definition 2.54 (Seperating points)**.** An integral scheme (X, O_X) is said to seperate points if for all $x, y \in X$, there exists $f \in K(X)$ with

$$f \in \mathfrak{m}_x \subset O_{X,x} \subset K(X) \text{ and } f \notin \mathfrak{m}_y \subset O_{X,y}.$$

Remark 2.55. This definition is non-standard, and suffices for our one dimensional purposes. On the schemes we will be considering, this can be shown to be equivalent to X being seperated over Spec($\mathbb{Z}$) using the valuative criterion for seperatedness, Theorem 4.3 in [Har77].

Now we come to the definition of the objects we will be principally interested in, schemes built out of the spectra of Dedekind domains.

Definition 2.56 (Dedekind schemes)**.** A Dedekind scheme is a Noetherian integral scheme X that seperates points, such that X admits an open cover of affine schemes $\{U_i\}_{i\in I}$, such that

$$(U_i, O_X|_{U_i}) \cong (\mathrm{Spec}(A_i), O_{A_i})$$

where each A_i is a Dedekind domain.

These schemes will have a more palatable topology than arbitrary schemes, their proper closed subsets are the finite subsets of closed points of X.

Proposition 2.57. *For X a Dedekind scheme, the underlying topological space of X has nontrivial closed sets precisely the finite unions of closed points.*

Proof. We know X has an affine open cover $\{U_i\}_{i\in I}$, and we have a descending chain of closed subschemes

$$\bigcap_{i=1}^{k} X \setminus U_i.$$

By the Noetherian property, we have that X is covered by finitely many U_i. From characterisation *ii*) of Theorem 2.48 , on each affine subscheme U_i we see that the nontrivial closed sets are finite unions of closed points. □

We also have that open subschemes of Dedekind schemes are Dedekind schemes.

Proposition 2.58. *For U an open subset of X a Dedekind scheme, the scheme $(U, O_X|_U)$ is a Dedekind scheme.*

Proof. First, we show that a nonempty distinguished open subset of an affine Dedekind scheme is an affine Dedekind scheme. This translates to the statement that for R a Dedekind domain, $R[1/f]$ is also a Dedekind domain. This ring is Noetherian, and integrally closed, so we just need to check that it is one dimensional. For this, we have that R is a domain that isn't a field, so has infinitely many prime ideals, since f is contained in finitely many prime ideals, we have that $R[1/f]$ is also one dimensional.

So now an arbitrary open subscheme of a Dedekind scheme is a union of distinguished opens on an affine cover, so it is a Dedekind scheme. □

The fact that Dedekind schemes are one dimensional allows the closed points of X to see more information about X than closed points do for arbirary schemes, and allows for a more "point centric" viewpoint. The next definition is a manifestation of this, we can study X using valuations on its field of fractions $K(X)$.

Proposition 2.59. *For any closed point $P \in X$, the stalk $O_{X,P}$ is a discrete valuation ring, and the discrete valuation v_P on $O_{X,P}$ extends to a discrete valuation on $K(X)$ by $v_P(a/b) = v_P(a) - v_P(b)$. The subring associated to v_P in $K(X)$ is precisely $O_{X,P}$.*

Proof. The stalk at P can be calculated with reference to an affine U containing P. Letting U be isomorphic to $\mathrm{Spec}(A)$ for A a Dedekind domain, the claim follows at once, since it holds in the affine case, and $\mathrm{Quot}(O_X(U)) = K(X)$. □

Since we have a valuation for each point P of a Dedekind scheme X, the field $K(X)$ comes with an abundance of natural valuations. We should think of the valuation v_P as an indicator of the order of vanishing of a function $f \in K(X)$ at the point P.

Since the associated ring O_{v_P} is just the stalk of O_X at P, so we may view the stalk at P as those functions on X which have all their "poles" away from P.

For a Dedekind scheme X, we may construct a ringed space from the collection of valuations v_P.

Construction 2.60. Let $K(X)$ be the field of fractions of a Dedekind scheme X. We define a locally ringed space $(v(X), \tilde{O}_X)$ as follows.

- The points of $v(X)$ are equivalence classes of valuations v on $K(X)$ such that O_v contains some $O_X(U)$ for U an affine open subset of X.
- The topology is generated by setting every nonzero valuation to be closed.
- The sheaf of rings has sections over V given by

$$\tilde{O}_X(V) = \bigcap_{v \in V} O_v = \{x \in K(X) | v(x) \geq 0 \text{ for all } v \in V\}.$$

Remark 2.61. One may note that there is a unique surjective valuation in each equivalence class of valuations, and indeed we could have defined this space as the set of surjective valuations, without mentioning equivalence classes. This makes the future functoriality of this construction less elegant however, so we decided against this.

The canonical maps $O_X(U) \to K(X)$ realise O_X as a subsheaf of the constant $K(X)$ sheaf. For U open in X, we have $\tilde{O}_X(U) \subset O_X(U) \subset K(X)$, since a function $f \in \tilde{O}_X(U)$ with no poles on U restricts to a compatible family of functions on an affine cover U_i of U, which by the sheaf condition glues to an element in $O_X(U)$.

Theorem 2.62. *The map of locally ringed spaces*

$$(X, O_X) \to (v(X), \tilde{O}_X)$$

given by taking each point P of X to its associated valuation v_P, and including $\tilde{O}_X(U)$ into $O_X(U)$ is an isomorphism of locally ringed spaces.

Proof. First, on the underlying topological space, this map is a homeomorphism. For injectivity, we have that for any two closed points P, Q, there is some $f \in K(X)$ such that $v_P(f) \neq v_Q(f)$, by our seperatedness assumption, and the generic point is the unique point mapped to the trivial valuation.

For surjectivity, note that if $v \in v(X)$ is a nontrivial valuation, then the associated ring O_v contains some $O_X(U)$ for U affine, so the subset of $O_X(U)$ with $v(x) > 0$ is a prime ideal $\mathfrak{p}$ of $O_X(U)$, with valuation v_P equivalent to v. Both sets have the topology generated by setting all points closed, except the respective exceptional points, so this map is a homeomorphism.

It remains to show that the map of structure sheaves is an isomorphism. We therefore need to check that if $f \in O_X(U)$, then $v_P(f) \geq 0$ for all $P \in U$. Passing to an affine U' containing P, we see that $f \in O_X(U')$, so $f \in O_{X,P}$, which is just the local ring $(O_X|_{U'})_P$. Since the valuation on this ring is v_P, we see $v_P(f) \geq 0$. □

The upshot of this description is that we can directly access the structure sheaf of a Dedekind scheme in terms of valuations. We may think of points of Dedekind schemes as points of a space, as prime ideals in a family of rings, or as valuations on a field. From here, we will identify these objects, and freely switch between intepretations. For a point P in X, we denote the corresponding valuation by v_P, and the prime ideal of $O_{X,P}$ by $\mathfrak{p}$.

2.6 The projective line

Let's now look at a fundamental example of a Dedekind scheme that is not affine, the projective line. Let k be a field, and consider the topological space

$$\mathbb{P}^1_k := \mathrm{Spec}(k[t]) \cup \{\infty\}$$

where we topologise this by setting $\{\infty\}$ to be closed, and setting the closure of $x_{gen} \in \mathrm{Spec}(k[t])$ to be all of $\mathbb{P}^1_k$.

We notice that the points of $\mathbb{P}^1_k$ correspond to valuations associated to prime ideals on $k[t]$, with one extra valuation v_∞, given by

$$v_\infty(\frac{f}{g}) := \deg(g) - \deg(f).$$

These are all valuations on $k(t)$, and we give the sheaf of rings by the same process as Construction 2.60

$$O_{\mathbb{P}^1_k}(U) = \bigcap_{v \in U} O_v = \{f \in k(t) | v(f) \geq 0 \text{ for all } v \in U\}.$$

Proposition 2.63. *The ringed space $(\mathbb{P}^1_k, O_{\mathbb{P}^1_k})$ is a Dedekind scheme.*

Proof. Let's check that we have an open cover of $\mathbb{P}^1_k$ by affine Dedekind schemes. For this, consider the subsets $\mathbb{P}^1_k \setminus v_\infty$ and $\mathbb{P}^1_k \setminus v_0$, where v_0 is the point associated to the ideal $(t) \subset k[t]$. These open sets cover $\mathbb{P}^1_k$, and we may recognise the sub-locally ringed spaces as $\mathrm{Spec}(k[t])$ and $\mathrm{Spec}(k[t^{-1}])$ respectively. These are both the spectra of principal ideal domains, hence Dedekind domains.

We see directly that $\mathbb{P}^1_k$ is a Noetherian scheme, and we may seperate any two points with an appropriately chosen rational function. □

Note, we could have described $\mathbb{P}^1_k$ equally well as $\mathrm{Spec}(k[t])$ and $\mathrm{Spec}(k[t^{-1}])$ glued along the respective open inclusions of $\mathrm{Spec}(k[t, t^{-1}])$, but since we haven't developed the theory of glueing locally ringed spaces, we will have to content ourselves with this somewhat ad hoc description.

The example of $\mathbb{P}^1_k$ is instructive for visualising Dedekind schemes in general, so let's try to describe it geometrically. Let's look over $\mathbb{C}$, and try visualising $\mathrm{Spec}(\mathbb{C}[t])$ first. Its closed points are the prime ideals of $\mathbb{C}[t]$, so since every nonzero prime ideal is $(t - z)$ for some $z \in \mathbb{C}$, identifying $z \in \mathbb{C}$ with $(t - z)$, we have the complex plane, with the cofinite topology. We have one extra point however, which has closure the whole plane. We can think of this geometrically as the "bulk" of the complex plane. Since this point is the intersection of all open sets, it doesn't contain any points of $\mathbb{C}$, but by removing finitely many points, we can't get rid of the "bulk" of $\mathbb{C}$.

To visualise $\mathbb{P}^1_{\mathbb{C}}$, we can take the complex plane, and add one point v_∞ "at infinity", to form the Riemann sphere, which is homeomorphic to S^2 if we take the usual Euclidean topology. Of course, our topology is much coarser, but thinking of $\mathbb{P}^1_{\mathbb{C}}$ as the Riemann sphere with fewer open sets is a good heuristic to aid understanding. For instance, our sheaf of rings has

$$O_{\mathbb{P}^1_{\mathbb{C}}}(\mathbb{P}^1_{\mathbb{C}} - \{P_1, ..P_n\}) = \bigcap_{Q \in \mathbb{P}^1_{\mathbb{C}} - \{P_1,..P_n\}} O_Q.$$

These are precisely the rational functions in t with poles only at $P_1, ..P_n$.
We may also note that the the stalk of $O_{\mathbb{P}^1_{\mathbb{C}}}$ at the point $(t - \alpha)$ is given by those rational functions $f(t)/g(t)$ with $g(t)$ coprime to $t - \alpha$, that is, with $g(\alpha) \neq 0$.
Since the only rational functions without poles are constants, we see that

$$O_{\mathbb{P}^1_{\mathbb{C}}}(\mathbb{P}^1_{\mathbb{C}}) = k.$$

So we see that the Dedekind scheme $\mathbb{P}^1_{\mathbb{C}}$ is not affine.

Remark 2.64. We will not be using any complex analysis in this thesis, but the behaviour of meromorphic functions on Riemann surfaces will provide helpful intuition for the algebraic properties of Dedekind schemes. Many of our results have an exact analytic counterpart in the setting of Riemann surfaces.

Chapter 3

Dedekind schemes

3.1 Finite morphisms and integral closures

In this section we will be considering a specific class of morphisms between Dedekind schemes, and the algebraic invariants they entail.

Definition 3.1 (Finite morphisms)**.** A morphism of schemes $f : X \to Y$ is finite if $O_X(f^{-1}(U))$ is a finitely generated $O_Y(U)$-module, for all open U, and there exists an affine cover U_i of Y such that $f^{-1}(U_i)$ is affine for each i.

This finiteness allows us to define an associated finite degree extension of fraction fields.

Proposition 3.2. *If $f : X \to Y$ is a finite morphism of Dedekind schemes, then f maps the generic point of X to the generic point of Y.*

Proof. Let U be an open affine of Y such at $f^{-1}(U)$ is an open affine in X. Now note that both of these rings are Dedekind domains, so if $O_Y(U) \to O_X(f^{-1}(U))$ wasn't injective, then its image would be a field, and $O_X(f^{-1}(U))$ would therefore be a field since it is finitely generated as a module over $O_X(U)$. This constradicts that $f^{-1}(U)$ is affine, since $O_X(f^{-1}(U))$ is a Dedekind domain. Thus, the map is injective, so the preimage of the zero ideal in $O_X(f^{-1}(U))$ is the zero ideal in $O_Y(U)$, so the map f maps x_{gen} to y_{gen}. □

Passing to stalks, we obtain a field extension $K(X)/K(Y)$ associated to a finite morphism of Dedekind schemes.

Definition 3.3 (Degree of a finite morphism)**.** For a finite morphism $f : X \to Y$ of Dedekind schemes, we say the degree of f is the degree of the field extension $K(X)/K(Y)$.

Note that the degree is a finite integer, since for affine subsets $U \subset Y$, $f^{-1}(U) \subset X$, we have $O_X(f^{-1}(U))$ is a finitely generated module over $O_Y(U)$, so its field of fractions has a finite $\mathrm{Quot}(O_Y(U))$ basis.

We will be using the algebraic notions of integral elements, integral ring extensions, and integral closures, so the reader is invited to consult chapter 5 of [AM69] for definitions and their basic properties.

We will supply a proof of the following proposition however, as it is critical for understanding the relation between finiteness and integrality.

Proposition 3.4. *Let A be a domain. If B is an A algebra, which is finitely generated as an A-module, then B is integral over A.*

Proof. Picking a finite A-module generating set $\{s_i\}_{i=1}^n$ of B, we may describe the effect of multiplication by any x in B on these generators as an A valued matrix μ_x. Viewing this as a matrix with coefficients in $\mathrm{Quot}(A)$, by the Cayley Hamilton theorem of linear algebra, we see that $P(\mu_x) = 0$, where P is the monic characteristic polynomial of the matrix μ_x. Since the entries of μ_x had coefficients in A, P has coefficients in A. This gives that the action of $P(x)$ on B is zero, so $P(x) \cdot 1 = 0$, so $P(x) = 0$, and x is integral over A. □

Remark 3.5. This lemma is true without the assumption that A be a domain, using the generalisation of the Cayley Hamilton theorem that holds for matrices valued in any commutative ring.

Corollary 3.6. *If $f : X \to Y$ is a finite morphism of Dedekind schemes, then for all U, we have that $O_X(f^{-1}(U))$ is the integral closure of $O_Y(U)$ in $K(X)$.*

Proof. By the previous proposition, we see that $O_X(f^{-1}(U))$ is integral over $O_Y(U)$, and by the description of Theorem 2.62, we see that for any element $x \in K(X)$ not contained in $O_X(f^{-1}(U))$, there exists some equivalence class of valuations w extending a class of valuations v on $K(Y)$, such that $w(x) < 0$. So if $x^n = \sum_{i=0}^{n-1} a_i x^i$ is an equation witnessing the integrality of x over $O_Y(U)$, then applying a representative w of the valuation class to both sides yields the desired contradiction, since w is positive on A. □

From this corollary we see that a finite morphism $f : X \to Y$ is totally determined by its induced finite degree extension of fields of fractions. This is to say, the functor from Dedekind schemes with finite maps to field extensions is faithful. We may interpret corollary 3.6 as a recipe for constructing a Dedekind scheme. Given a finite extension of function fields we may glue the spectra of the integral closures of $O_X(U_i)$ in L for $\{U_i\}_{i\in I}$ an affine open cover of X. So starting from a finite degree extension $K(X) \xrightarrow{\iota} L$, we have the following construction of a locally ringed space. We say a valuation $v : L \to \mathbb{Z}$ extends a valuation w on $K(X)$ if $v \circ \iota = w$.

Construction 3.7. Let $L/K(X)$ be a fixed, finite extension of $K(X)$ for X a Dedekind scheme. We define a locally ringed space $(v_X(L), O_{v_X(L)})$ as follows:

- The points of $v_X(L)$ are the equivalence classes of valuations that extend those valuations of $K(X)$ associated to points of X.
- This space has topology generated by setting each equivalence class of nonzero valuations to be closed.
- This space has a sheaf of rings given by

$$O_{v_X(L)}(U) := \bigcap_{v\in U} O_v = \{x \in L | v(x) \geq 0 \text{ for all } v \in U\}.$$

This is a locally ringed space, since for any two distinct nonequivalent valuations v, w, we may find an $x \in L$ with $v(x) \geq 0$, and $w(x) < 0$, so the stalk at v is just O_v.

We have a natural map of locally ringed space $v_X(L) \xrightarrow{\iota} X$ given on points by composing equivalence classes of valuations with the map $K(X) \to L$, with the natural map induced from ι on structure sheaves.

Remark 3.8. We have suppressed the field extension $\iota : K(X) \to L$ in our definition here for notational clarity, but one should bear in mind that this construction does depend on the map ι. This construction is also the reason we chose to work with equivalence classes of valuations rather than surjective valuations in Construction 2.60.

Theorem 3.9 (Krull-Akizuki Theorem)**.** *If X is a Dedekind scheme, and $L/K(X)$ is a finite extension, then the locally ringed space $(v_X(L), O_{v_X(L)})$ is a Dedekind scheme.*

Proof. We will give the proof assuming that the morphism $v_X(L) \to X$ is finite, noting that this will always be the case for the Dedekind schemes we are principally interested in. For the general case, we refer to [Bou89].

We claim that the ring $O_{v_X(L)}(f^{-1}(U))$ is a Dedekind domain for all nonempty affine open subsets $U \subset X$. Let $B = O_{v_X(L)}(f^{-1}(U))$, and $A = O_X(U)$.

First, since B is a finitely generated module over the Noetherian A, it is Noetherian as a module over A, so is Noetherian as a ring, since $A \subset B$.

The ring B is also integrally closed, since any $x \in L$ integral over B has the A submodule $B[x] \subset L$ finitely generated over B, hence integral over A.

The ring B has Krull dimension 1. First, we need that any nonzero prime ideal $\mathfrak{q}$ of B has nonzero intersection with A. To see this, pick a nonzero element $x \in \mathfrak{q}$ and consider a minimal monic polynomial witnessing its integrality. Its constant term will be nonzero, and contained in A.

Therefore $B/\mathfrak{q}$ is a finite dimensional domain over the field $A/\mathfrak{p}$, so is itself a field. So every nonzero prime ideal is maximal, and B is not a field since it contains an element x of A, and a valuation v on L such that $v(x) > 0$.

Thus, B is a Dedekind domain. We may note from Theorem 2.62 that the restriction of $(v_X(L), O_{v_X(L)})$ to $f^{-1}(U)$ as is given as $\operatorname{Spec}(O_{v_X(L)}(f^{-1}(U)))$, giving the result. $\square$

Unfortunately, the map $v_X(L) \to X$ is not necessarily finite in full generality.

This construction is used to construct the Dedekind schemes considered in algebraic number theory. For any finite extension K of $\mathbb{Q}$, we have the canonical map $\mathbb{Q} \to K$. Taking the integral closure of $\mathbb{Z}$ in K yields the a ring, traditionally called the ring of integers of K, denoted O_K.

The associated Dedekind schemes $\operatorname{Spec}(O_K)$ are all affine, and so are traditionally just treated as rings. The upshot of this alternate viewpoint is that it allows for a clear distinction between the algebra-geometric properties of O_K, and the purely number theoretic properties.

The principal "number theoretic" property of the rings O_K is the fact that their residue fields at maximal ideals are all finite. Viewing these $\operatorname{Spec}(O_K)$ as affine Dedekind schemes with this finiteness property leads one to the incredibly fruitful analogy between number fields and curves over a finite field.

3.2 The trace and the norm

For any finite field extension L/K, we may view elements of L as K-linear maps on L via multiplication, $\mu_x(y) = xy$. This gives rise to two fundamental maps from L to K, the norm and the trace.

Definition 3.10. The field norm and trace of x are defined by taking the determinant and trace of the K linear map μ_x.

$$\mathrm{N}_{L/K}(x) = \det(\mu_x)$$

$$\mathrm{Tr}_{L/K}(x) = \mathrm{Tr}(\mu_x)$$

Note that the norm and trace maps are multiplicative and additive respectively, and that the trace is K-linear.

In a Galois extension L/K, the norm and trace of an element x are given by the product or sum of the Galois conjugates of x, viewed as an element of K. To see this, recall from field theory that the eigenvalues of μ_x in an algebraic closure of L are the images of x under the $[L : K]$ distinct field embeddings into an algebraic closure of L.

Both of these maps are fundamental in the context of Dedekind schemes, but we will principally concerned with the trace map, and the bilinear form it yields.

Definition 3.11 (The trace form)**.** The trace form of L/K is the symmetric linear map $\langle\ ,\ \rangle : L \otimes_K L \to K$ given by

$$\langle x, y\rangle = \mathrm{Tr}_{L/K}(xy).$$

Proposition 3.12. *As a K-valued symmetric form on L, the trace form is either nondegenerate, or identically zero.*

Proof. If the trace form is degenerate, for some nonzero $x \in L$, we have $\langle x, y\rangle = 0$ for all $y \in L$, so letting $y = x^{-1}z$, we see $\mathrm{Tr}(z) = 0$ for all $z \in L$. □

Definition 3.13. A finite morphism of Dedekind schemes $f : X \to Y$ is separable if the induced field extension $K(Y) \to K(X)$ is separable.

For us, the trace form is a fundamental invariant of the finite field extensions we will be considering, so let us recall the following fact from field theory, whose proof may be found as theorem 5.2 of [Lan02].

Proposition 3.14. *A finite degree field extension is separable if and only if the associated trace form is nondegenerate.*

The presence of a nondegenerate trace form has many consequences, but first, we have the following compatibility result.

Proposition 3.15. *Let A be an integrally closed domain, with fraction field K. If L/K is a finite field extension, and B is the integral closure of A in L, then for any x in B, the field trace $\mathrm{Tr}_{L/K}(x)$ is in A.*

Proof. If f is not separable, then the claim is trivial since the trace form is identically zero, so assume the extension is separable. In that case, let M denote a Galois closure of L/K. For x in B, x is integral over A by assumption. By applying elements of the Galois group to a monic polynomial witnessing this, we see its Galois conjugates in M are also integral over A.

So the sum of these conjugates is integral over A, so the trace of x is integral over A, and contained in K. Hence it is contained in A since this ring is integrally closed. □

In a separable finite extension $L/K(X)$, we may use the trace form to prove that the map from Construction 3.7 is finite.

Theorem 3.16. *For $L/K(X)$ a separable field extension the morphism $v_X(L) \to X$ of Construction 3.7 is finite.*

Proof. Construction 3.7 produces affine schemes from affine schemes, so it suffices to prove the following finiteness statement. Let B/A be rings such that A is Noetherian, contained in a separable field extension L/K with $L = \text{Quot}(B)$, $K = \text{Quot}(A)$, with B the integral closure of A. Then B is finitely generated as an A-module.

Since the trace of L/K is nondegenerate and symmetric, pick an orthogonal K-basis $\{b_j\}_{j=1}^n$ of L, scaling to ensure that it lives in B. We claim that every element of B is contained in A submodule of L spanned by $\frac{b_j}{d}$ where d is given by

$$d := \prod_{j=1}^{n} \text{Tr}(b_j^2).$$

Given an element x of B, we can express x in the b_j basis as $x = \sum_{j=1}^n \alpha_j b_j$. By Proposition 3.15, $\text{Tr}(b_j x) \in A$, and the b_j are orthogonal, so $\alpha_j \text{Tr}(b_j^2) \in A$, so

$$\alpha_j b_j = \alpha_i \text{Tr}(b_i^2) \prod_{j \neq i} \text{Tr}(b_i^2) \frac{b_j}{d}.$$

So x lies in the A span of the $\frac{b_j}{d}$, and so B is finitely generated since A is Noetherian. □

For a finite morphism $f : X \to Y$ of Dedekind schemes, the associated morphism of sheaves of rings $O_Y \to f_* O_X$ induces, for each U open in X, an $O_Y(U)$-module structure on $O_X(f^{-1}(U))$. We will want to view this collection as a compatible family of modules over the rings $O_X(U)$, leading to the following definition.

Definition 3.17 (Sheaf of O_X-modules on a scheme)**.** A sheaf $\mathcal{F}$ of O_X-modules over X is a sheaf $\mathcal{F}$ of abelian groups on X, such that each $\mathcal{F}(U)$ is an $O_X(U)$-module, such that if $U \subset V$, and we view $\mathcal{F}(U)$ as an $O_X(V)$-module by pullback, the restriction maps $\mathcal{F}(V) \to \mathcal{F}(U)$ are morphisms of $O_X(V)$-modules.

A morphism of sheaves of O_X-modules is a morphism of sheaves of abelian groups such that over each open U we have a morphism of $O_X(U)$-modules.

We will investigate sheaves of O_X-modules further in chapter 4. For now we are principally interested in upgrading the trace map to a map of sheaves.

Proposition 3.18 (Trace is a sheaf morphism)**.** *If $f : X \to Y$ is a finite morphism of Dedekind schemes, then the restriction of the field trace*

$$\mathrm{Tr}_{K(X)/K(Y)} : K(X) \to K(Y)$$

induces a morphism of O_Y-modules

$$\mathrm{Tr}_{X/Y} : f_*O_X \to O_Y.$$

Proof. Since $f_*O_X(U)$ is the integral closure of $O_Y(U)$ in $K(X)$, by Proposition 3.15 we have that $\mathrm{Tr}_{K(X)/K(Y)}$ descends to a map of these subrings.

This is a morphism of O_Y-modules since the trace is natural, and is $K(Y)$-linear. □

3.3 Completion

Before investigating these morphisms further, we need to take a small detour, to review some commutative algebra, which plays a key role in this chapter.

Definition 3.19 (Completion)**.** Given an ideal I in a ring R, we have a family of natural maps $R/I^{n+1} \to R/I^n$. We define the I-adic completion of R to be the ring $\widehat{R}^I := \lim(R/I^n)$. Similarly we define the I-adic completion of an R-module M to be $\widehat{M}^I := \lim M/I^nM$, a module over $\widehat{R}^I$.

We define the completion of O_X at the closed point P to be the $\mathfrak{p}$-adic completion of $O_{X,P}$, where $\mathfrak{p}$ is the maximal ideal corresponding to P.

When the ideal I is clear from context, we will drop the I, and refer to these completed objects as $\widehat{R}$ and $\widehat{M}$.

To intepret completion geometrically, recall the correspondence between closed subschemes and ideals of an affine scheme given by definition 2.34. Completion at the closed subscheme $S \subset X$ can be thought of as an "ultra-local" analysis of the functions defined on that subscheme S. When we localise, we look at functions on X, and identify them based on their behaviour in an open neighbourhood of S. This can be thought of as "shrinking" the domain of the functions to be on smaller and smaller neighbourhoods of S. This results in a local ring, but still retains the information of the global functions, as the fraction field is the same.

Let's now consider the case when S is a closed point P. If we consider a function f on X, defined at P, we have a sequence of infinitesimal data associated to it, the cosets of f in $O_{X,P}/\mathfrak{p}^n$. For instance, the coset of f in $O_{X,P}/\mathfrak{p}$ is the "value" at $\mathfrak{p}$, and the coset in $O_{X,P}/\mathfrak{p}^2$, is this value plus a "cotangent" vector, corresponding to the lift of $O_{X,P}/\mathfrak{p}^2 \to O_{X,P}/\mathfrak{p}$. The completion at P is the ring of all collections of "infinitesmal data" at P, irrespective of whether they arise from a function on X actually defined at P. This is in a sense an "outward" description of the functions, which results in even more "locality" than localisation.

Example 3.20. Let's consider the affine Dedekind scheme $\operatorname{Spec}(\mathbb{C}[t])$, and the point $P = 0$, corresponding the ideal (t). Localising at $P = 0$ results in the stalk $\mathbb{C}[t]_{(t)}$, the rational functions without t dividing their denominator. Completing at $P = 0$ results in $\lim \mathbb{C}[t]/t^k \cong \mathbb{C}[[t]]$, and the natural composite map

$$\mathbb{C}[t]_{(t)} \to \widehat{\mathbb{C}}[t]_{(t)} \cong \mathbb{C}[[t]]$$

interprets a rational function as its sequence of Taylor coefficients at $P = 0$.

Proposition 3.21. *The completion of an R-module M with respect to I^n is canonically isomorphic to the completion with respect to I.*

Proof. The diagram of R/I^{ni} is naturally a final subfunctor of the diagram of R/I^n, so the inclusion induces an isomorphism of limits. □

The following lemma should be thought of as algebraic justification for the intepretion of completion as an ultra-local process.

Proposition 3.22 (Completion is ultra-local)**.** *For pairwise relatively prime ideals $\mathfrak{p}_i$ of R, and any ideal $I = \prod_{i=1}^n \mathfrak{p}_i^{e_i}$ the I-adic completion of an R-module M is isomorphic to the product of the $\mathfrak{p}_i$-adic completions of M. If M was itself an R algebra, then this decomposition is as R algebras.*

Proof. The isomorphism of the Chinese remainder theorem is natural, in that the following diagram commutes.

$$\begin{array}{ccc} M/I^{m+1}M & \longrightarrow & M/I^m M \\ \downarrow \sim & & \downarrow \sim \\ \prod_{i=1}^n M/\mathfrak{p}_i^{m+1}M & \longrightarrow & \prod_{i=1}^n M/\mathfrak{p}_i^m M \end{array}$$

So we have an induced isomorphism on the associated limits, which followed by the isomorphism of Proposition 3.21, gives the result. The isomorphism of the Chinese remainder theorem also preserves the R algebra structure of M, giving the second claim. □

Interpreting this geometrically, consider a finite subset S of an affine Dedekind scheme X. When we localise at S, the points are still "connected" in the sense that the fraction field detects more than just the disjoint collection of local rings $O_{X,P}$. When we complete at S however, we instead get the product of the completions at each point, the behaviour is truly "local".

Proposition 3.23. *The completion of a free R-module M is a free $\widehat{R}$-module $\widehat{M}$.*

Proof. A basis of M over R gives a compatible basis of each quotient, and thus a basis of $\widehat{M}$. □

Given a discrete valuation ring R, we may complete at its maximal ideal, and this process is functorial with respect to local morphisms of discrete valuation rings. Going "ultra-local" in this way results in an new extension of complete discrete valuation rings, cutting out unwanted global behaviour.

Theorem 3.24 (Completing discrete valuation rings)**.** *If R is a discrete valuation ring with maximal ideal $\mathfrak{m}$, then the $\mathfrak{m}$-adic completion of R is also a discrete valuation ring, and the ideal $\widehat{\mathfrak{m}}$ can be generated by any uniformising parameter of R.*

This theorem follows at once from the following lemma, which has the same hypotheses as the theorem.

Lemma 3.25. *For a fixed uniformising parameter π in R, any element α in $\widehat{R}$ can be expressed uniquely as $\pi^k u = \alpha$ for some unit $u \in \widehat{R}^*$.*

Proof. First, identify the elements of $\widehat{R}$ with sequences (α_i) such that $\alpha_i \in R/\mathfrak{m}^i$, and

$$\alpha_i \equiv \alpha_j \bmod \mathfrak{m}^k \text{ for } k \leq \min(i, j).$$

Given (α_i) in $\widehat{R}$, pick the minimal i such that α_i is nonzero in $R/\mathfrak{m}^{i+1}$. Let a_j be a lift of α_j to R, and express $a_j = \pi^i v_j$ for $v_j \in R^*$.

We claim that the reductions $\bar{v}_j$-mod $\mathfrak{m}^{j-i}$ of these v_j form a compatible sequence, and hence an element of $\widehat{R}$.

Let's check that $\bar{v}_l - \bar{v}_m = 0$-mod $\mathfrak{m}^{k-i}$ for $k \leq l, m$. From our intitial compatibility assumption, we see that $\pi^i(v_l - v_m) \equiv 0 \bmod \mathfrak{m}^k$, so we have $v_l - v_m \in \mathfrak{m}^{k-i}$ within R, from which reducing mod $\mathfrak{m}^{k-i}$ gives the result.

It remains to show that this element $(\bar{v}_j)$ is a unit in $\widehat{R}$. By minimality of i, we see that each v_j is a unit in R, and the cosets of their inverses are compatible. □

From here, we will be considering extensions of rings at varying degrees of locality, so we will introduce some nonstandard notation for clarity.

Let $\phi : A \to B$ be an injective map of rings, which we denote by B/A.

- We say B/A is local if A and B are local rings, and ϕ is a local ring morphism.
- We say B/A is semi-local if A is local, and B has finitely many prime ideals.
- We say B/A is global otherwise.

The source of this terminology is that for a finite map of Dedekind schemes $f : X \to Y$, we have a family of injective ring morphisms $O_Y(U) \to f_* O_X(U)$. The associated map on stalks at P in Y gives $O_{Y,P} \to (f_* O_X)_P$, a semilocal ring morphism. Picking a point Q mapping to P, we may compose this with the natural map to the stalk of Q to get a local extension $O_{Y,P} \to O_{X,Q}$.

From the previous theorem, we see that completion at a prime ideal preserves local morphisms for the class of rings we are interested in.

Lemma 3.26 (Completion in the semilocal setting)**.** *Let B/A be a semilocal ring morphism, with A a discrete valuation ring, such that B is a finite free A-module. The the completion $\widehat{B}^{\mathfrak{m}}$ of B with respect to the maximal ideal $\mathfrak{m}$ of A is free as an $\widehat{A}^{\mathfrak{m}}$ algebra.*

Proof. This follows at once from propositions 3.23 and 3.22. □

With reference to a finite morphism f, we will use the notation $Q : Q \to P$ to qualify over the points Q such that $f(Q) = P$.

Corollary 3.27. *For a finite morphism $f : X \to Y$ of Dedekind schemes, and point $Q \in X$, with $f(Q) = P \in Y$,the local ring morphism $\widehat{O}^{\mathfrak{p}}_{Y,P} \to \widehat{O}^{\mathfrak{q}}_{X,Q}$ gives $\widehat{O}^{\mathfrak{q}}_{X,Q}$ the structure of a finite* free *$\widehat{O}^{\mathfrak{p}}_{Y,P}$ algebra. Furthermore the semilocal completion decomposes as a product of the local completions*

$$\widehat{(f_*O_X)}^{\mathfrak{p}}_P \cong \prod_{Q:Q\to P} \widehat{O}^{\mathfrak{q}}_{X,Q}.$$

Proof. This follows at once from Lemma 3.26 and Proposition 3.22. □

This corollary is in some sense the whole point of this section, as the local morphism $O_{Y,P} \to O_{X,Q}$ will almost never give a free $O_{Y,P}$-module structure to $O_{X,Q}$, but by completing, we can ensure this.

The upshot of this is that we can now localise our linear algebraic constructions, the trace and the norm map.

For a finite morphism $f : X \to Y$ of Dedekind schemes, we use the notation Q/P to refer to a pair of points $Q \in X$, $P \in Y$, such that $f(Q) = P$. Let $f : X \to Y$ be a finite morphism of Dedekind schemes.

Definition 3.28. The semilocal trace at P is the map

$$\mathrm{Tr}_{X/P} : (f_*O_X)_P \to O_{Y,P}$$

obtained by taking the stalk at P of the trace map $\mathrm{Tr}_{X/Y}$.

Completing with respect to $\mathfrak{p}$, and noting the diagonal decomposition of corollary 3.27, we define $\mathrm{Tr}_{Q/P}$, the local trace at Q/P to be the following composite.

$$\begin{array}{ccc} \widehat{O}^{\mathfrak{q}}_{X,Q} & \longrightarrow & \widehat{(f_*O_X)}^{\mathfrak{p}}_P \\ & {\scriptstyle \mathrm{Tr}_{Q/P}} \searrow & \downarrow {\scriptstyle \widehat{\mathrm{Tr}}_{X/P}} \\ & & \widehat{O_{Y,P}}^{\mathfrak{p}} \end{array}$$

This is alternatively given by viewing an element x of $\widehat{O}^{\mathfrak{q}}_{X,Q}$ as an endomorphism μ_x of $\widehat{(f_*O_X)}^{\mathfrak{p}}_P$ by left multiplication, and taking the trace of this endomorphism.

We define the semilocal norm $\mathrm{N}_{X/P}$ of x by taking the determinant of μ_x, and the local norm $\mathrm{N}_{Q/P}$ by taking the determinant of μ_x, this time viewing it as an $\widehat{O}^{\mathfrak{p}}_{Y,P}$ linear endomorphism of the free module $\widehat{O}^{\mathfrak{q}}_{X,Q}$.

This results in the following decomposition of the semilocal norm and trace.

Proposition 3.29. *For all $x \in (f_* O_X)_P$, we have that the semilocal trace and norm are the sum and product of the local traces and norms repectively.*

$$\prod_{Q:Q\to P} \mathrm{N}_{Q/P} = \mathrm{N}_{X/P}$$

$$\sum_{Q:Q\to P} \mathrm{Tr}_{Q/P} = \mathrm{Tr}_{X/P}$$

Proof. The map $(f_* O_X)_P \to \widehat{(f_* O_X)}_P^{\mathfrak{p}}$ in view of the decomposition of corollary 3.27 maps x to $(x, x, .., x)$, giving the result by elementary linear algebra. □

3.4 Local invariants of a finite morphism

In this section, let $g : X \to Y$ be a fixed finite morphism of Dedekind schemes. All discussion is with reference to this fixed morphism unless otherwise specified. For each pair of points Q/P, we can associate a number of invariants. These invariants are local in the sense of depending only on the local extension $O_{Y,P} \to O_{X,Q}$.

We will use the notation $\kappa(P)$ to denote the residue field of the local ring $O_{X,P}$, that is, $\kappa(P) := O_{X,P}/\mathfrak{p}$.

Definition 3.30. The residual degree $f_{Q/P}$ at Q/P is the degree of the extension $\kappa(Q)/\kappa(P)$.

Note that since our morphism is finite, this is a well defined positive integer. Our other primary local invariant is the ramification index.

Definition 3.31. The ramification index $e_{Q/P}$ is the unique integer such that $\mathfrak{p} = \mathfrak{q}^{e_{Q/P}}$ as $O_{X,Q}$ ideals.

We will abuse our notation slightly and sometimes refer to these as $f_{\mathfrak{q}/\mathfrak{p}}$ and $e_{\mathfrak{q}/\mathfrak{p}}$ when we wish to think of the points Q, P as their corresponding prime ideals.

To show that this is well defined, we will use the following useful lemma, showing we may extend proper ideals to proper ideals.

Lemma 3.32 (Going up). *If B/A is an extension of domains, with B a finitely generated A-module, then any proper ideal I in A remains a proper ideal once extended to B.*

Proof. It suffices to show the claim for elements, that if x is not invertible in A, then x is not invertible in B. If x had an inverse x^{-1} in B, then since the extension

B/A is finite, x^{-1} is integral over A. Consider a monic polynomial witnessing the integrality of x^{-1} of minimal degree. We see that by multiplying by x, we obtain a polynomial in x^{-1} of strictly smaller degree, which is still monic. Thus, since x^{-1} is not in A, no such x^{-1} in B can exist. □

Corollary 3.33. *The integer $e_{Q/P}$ is well defined and nonzero.*

Definition 3.34 (Ramified points)**.** With respect to our finite morphism g, a point $Q \in X$ is ramified in X if either of the two conditions hold.

- $e_{Q/P} > 1$
- $\kappa(Q)/\kappa(P)$ is an inseparable field extension, where $P = f(Q)$.

A point P in Y is ramified in Y if some Q lying above it is ramified in X.

These local invariants respect composition of morphisms, if $X \to Y$ and $Y \to Z$ are finite morphisms, and Q/P and P/S are pairs of points, then

$$e_{Q/P}e_{P/S} = e_{Q/S} \text{ and } f_{Q/P}f_{P/S} = f_{Q/S}.$$

Both of these quantities are also invariant under completion, since they are both invariants of $O_{X,Q}/\mathfrak{p} \cong \widehat{O_{X,Q}}^{\mathfrak{p}}/\widehat{\mathfrak{p}}$.

Theorem 3.35 (Degree is seen semilocally)**.** *If the degree of the morphism g is n, then for any closed point P of Y, we have*

$$\sum_{Q:Q\to P} e_{Q/P}f_{Q/P} = n.$$

To prove this, we will first need the following lemma.

Lemma 3.36. *For B/A a local extension of discrete valuation rings with maximal ideals $\mathfrak{q}/\mathfrak{p}$, the dimension of $B/\mathfrak{p}$ as an $A/\mathfrak{p}$ vector space is $e_{\mathfrak{q}/\mathfrak{p}}f_{\mathfrak{q}/\mathfrak{p}}$.*

Proof. In view of the definition of $e_{\mathfrak{q}/\mathfrak{p}}$, by induction we will prove that the dimension of $B/\mathfrak{q}^m$ over $A/\mathfrak{p}$ is $m \cdot f_{\mathfrak{q}/\mathfrak{p}}$, for $m \leq e_{\mathfrak{q}/\mathfrak{p}}$. For any $1 \leq j \leq e_{\mathfrak{q}/\mathfrak{p}}$ we have a short exact sequence of $A/\mathfrak{p}$ vector spaces

$$0 \to \mathfrak{q}^{j-1}/\mathfrak{q}^j \to B/\mathfrak{q}^j \to B/\mathfrak{q}^{j-1} \to 0.$$

By picking a uniformising parameter π of B, we see that $\mathfrak{q}^{j-1}/\mathfrak{q}^j$ is one dimensional as a $B/\mathfrak{q}$ vector space, hence $f_{\mathfrak{q}/\mathfrak{p}}$ dimensional over $A/\mathfrak{p}$, so the result follows by induction. □

Proof of Theorem 3.35. First, note that the degree of g may be computed semi-locally, since the fraction fields of $(g_*O_X)_P$ and $O_{Y,P}$ are still $k(X)$ and $k(Y)$. Since $O_{Y,P}$ is a Dedekind domain, and the extension B/A is finite, by Lemma 3.32 we have that $\mathfrak{p}$ extends to a nonzero ideal in B, hence can be expressed as $\mathfrak{p} = \prod \mathfrak{q}_i^{e_{Q_i/P}}$. Complete both sides with respect to $\mathfrak{p}$, and observe the diagonal decomposition of corollary 3.27, and Lemma 3.36 together give the result. □

Lemma 3.32 also shows that finite morphisms of Dedekind schemes are surjective.

Proposition 3.37. *A finite map of Dedekind schemes $g : X \to Y$ is surjective and has finite fibres.*

Proof. Surjectivity on closed points follows from Lemma 3.32, surjectivity on the generic point is Proposition 3.2 and the finiteness of fibres follows from continuity, in view of the topology our Dedekind schemes have. □

Corollary 3.38. *Finite morphisms of Dedekind schemes are open maps on the underlying topological spaces.*

Proof. This follows from the topology of Dedekind schemes coupled with the previous proposition. □

Let's now consider the morphism $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ associated to the field extension

$$\begin{aligned} k(t) &\to k(x) \\ t &\to x^2. \end{aligned}$$

On the complement of ∞, we see that this is the ring morphism $\mathbb{C}[t] \to \mathbb{C}[x]$ which maps t to x^2. From this, we see that the ideal (t) becomes $(x)^2$, so (t) is ramified, with ramification index 2.

Under the bijection between closed points away from infinity and $\mathbb{C}$, we see that this map is given geometrically by $z \to z^2$ on the complex plane. Ramification is the phenomena of the different sheets of this map collapsing at a single point.

It is not hard to visualise the map $z \to z^n$ in a similar manner, and since ramification is a local phenomena, this is in some sense all there is to visualising it, in the intuitive case of $\mathbb{C}$.

3.5 Divisors on Dedekind schemes

We will be interested in describing the points of X and Y which are ramified, and for this we introduce a system for describing the points of a Dedekind scheme, with mutliplicities.

Definition 3.39 (Divisors)**.** The group of divisors $\mathrm{Div}(X)$ on a Dedekind scheme X is the free abelian group on the closed points of X. A divisor is effective if it is an $\mathbb{N}$ linear combination of points of X, and we have a partial order on divisors given by $D \geq D'$ if $D - D'$ is effective. A divisor D is negative if $-D$ is effective.

Definition 3.40. Let $D = \sum_{P \in X} a_P P$ be a divisor. The support of D is the subset of X of points such that $a_P \neq 0$. We define the order v_P of the divisor D at P to be $v_P(D) := a_P$.

These groups have natural restriction maps $\mathrm{Div}(X) \to \mathrm{Div}(U)$ given by taking projections, and this collection of data is seen to be a sheaf of abelian groups on the Dedekind scheme X.

We will be encountering divisors in many forms throughout this thesis, and we will learn progressively more interpretations of them.

In light of Theorem 2.48, for an affine Dedekind scheme $\mathrm{Spec}(R)$, we may identify effective divisors on X with nonzero ideals of R, or equivalently closed subschemes of $\mathrm{Spec}(R)$.

We have a natural source of divisors on a Dedekind schemes X, associated to nonzero elements of $K(X)$. For x in $K(X)^*$, we define the divisor $\mathrm{div}(x)$ by

$$\mathrm{div}(x) := \sum_{P \in X} v_P(x) P.$$

This sum is finite since $x = g/h$ for some g, h defined on an open affine U, so since g, h have poles contained entirely in the complement of U, and only finitely many zeros. Note that $v_P(\mathrm{div}(x)) = v_P(x)$.

From this, we may also view a divisor as giving prospective "orders of vanishing" at a finite set of points.

Definition 3.41. A divisor D is principal if it is of the form $\mathrm{div}(f)$ for some $f \in K(X)^*$.

Note that the principal divisors form a subgroup $P(X)$ of $\mathrm{Div}(X)$, since

$$\mathrm{div}(f) + \mathrm{div}(g) = \mathrm{div}(fg).$$

The difference between these groups is an important invariant of a Dedekind scheme, the class group.

Definition 3.42. The class group $\mathrm{Cl}(X)$ of a Dedekind scheme X is the quotient $\mathrm{Div}(X)/P(X)$.

We will investigate this group more thoroughly in chapter 4.

Remark 3.43. This group for the Dedekind scheme $\mathrm{Spec}(O_K)$ is called the ideal class group in algebraic number theory.

We have two natural operations on the groups of divisors induced from the finite map $g : X \to Y$.

Definition 3.44 (Pullback of divisors)**.** We define g^* on points by

$$g^*(P) = \sum_{Q:Q\to P} e_{Q/P}Q.$$

This extends by linearity to a map of abelian groups $\mathrm{Div}(Y) \xrightarrow{g^*} \mathrm{Div}(X)$.

Remark 3.45. In the affine case $A \to B$, where we can identify divisors with ideals, and this map is just given by extension of ideals. This is also true in the non affine case, with the correct notion of sheaves of ideals, given in definition 4.13.

Definition 3.46 (Pushforward of divisors)**.** We also have a natural pushforward on divisors, defined on points as

$$g_*(Q) = f_{Q/P}g(Q)$$

We extend this by linearity to a map $\mathrm{Div}(X) \xrightarrow{g_*} \mathrm{Div}(Y)$.

Remark 3.47. The following propositions can be seen as justification for the naturality of these definition, compared to taking the image and preimage without weighting by the local invariants.

Proposition 3.48 (Composition of pullback and pushforward)**.** *On* $\mathrm{Div}(Y)$*, the composition* g_*g^* *is given by multiplication by* n*, where* n *is the degree of the map* g*.*

Proof. This follows immediately from Theorem 3.35. □

The pushforward on divisors is also known as the norm map, because of the following compatibility.

Proposition 3.49 (Compatibility of the norm map and the pushforward)**.** *The norm map on divisors respects the usual norm on elements, in that for $x \in K(X)^*$, we have*

$$\mathrm{div}(\mathrm{N}_{K(X)/K(Y)}(x)) = g_*(\mathrm{div}(x)).$$

Proof. It suffices to show the claim locally at each pair Q/P since we have

$$v_P(\mathrm{N}_{K(X)/K(Y)}(x)) = v_P(\mathrm{N}_{X/P}(x)) = \sum_{Q:Q\to P} v_P(\mathrm{N}_{Q/P}).$$

Let $C = \widehat{O}^{\mathfrak{q}}_{X,Q}$, $A = \widehat{O}^{\mathfrak{p}}_{Y,P}$, and L/K be the associated field extension. We may split L/K further into the composition of L/L^I and L^I/K, where these are purely inseparable and separable respectively. Let B denote the integral closure of A in L^I, noting that C/B and B/A are both finite, as C is finite over A. Let the corresponding points be $Q/S/P$, and the associated maps be

$$\mathrm{Spec}(C) \xrightarrow{k} \mathrm{Spec}(B) \xrightarrow{h} \mathrm{Spec}(A).$$

Recalling the decomposition an arbitrary element given in Lemma 3.25, it suffices to prove the claim for a uniformising parameter of these rings.

For a purely inseparable extension of degree p^n, the norm is given by $\mathrm{N}(a) = a^{p^n}$, as can be seen from the characteristic polynomial of μ_a being $x^{p^n} - a^{p^n}$. Picking uniformisers π in B, γ in C, we have

$$\begin{aligned} e_{Q/S}\,\mathrm{div}(\mathrm{N}_{L/L^I}(\gamma)) = \mathrm{div}(\mathrm{N}_{L/L^I}(\pi)) &= \mathrm{div}(\pi^{p^n}) \\ &= p^n \cdot S \\ &= e_{Q/S} f_{Q/S} \cdot S \\ &= e_{Q/S} k_*(\mathrm{div}(\gamma)) \end{aligned}$$

For the separable extension L^I/K, we have that the norm is the sum over coset representatives of the fixed subgroup corresponding to L^I in a Galois closure. Since there exists a valuation w extending v_P, and any such valuation will have its conjugate valuations also extending v_P, we have

$$\mathrm{div}(\mathrm{N}_{L^I/K}(x)) = [G:H] v_P(x) \cdot P.$$

On the other side we have

$$h_*(\mathrm{div}(x)) = f_{S/P} e_{S/P} v_P(x) \cdot P.$$

Thus these are equal by Theorem 3.35.

The map $g_* \circ \operatorname{div}$ is functorial with respect to compositions, so it remains to show that

$$\operatorname{Div}(\mathrm{N}_{L^I/K} \circ \mathrm{N}_{L/L^I}(x)) = \operatorname{Div}(\mathrm{N}_{L/K}(x))$$

where L/L^I is purely inseparable, and L/K is separable. For this, recall from field theory [Lan02] that the norm in a purely inseparable extension is given by raising to the power of the degree p^n, and $\mathrm{N}_{L/K}(x) = \mathrm{N}_{L^I/K}(x)^{[L:L^I]}$ if $x \in L^I$. So to complete the proof, we compute:

$$p^n \operatorname{div}(\mathrm{N}_{L/K}(x)) = \operatorname{div}(\mathrm{N}_{L/K}(x^{p^n})) = \operatorname{div}(\mathrm{N}_{L^I/K}(x^{(p^n)^2})) = p^n \operatorname{div}(\mathrm{N}_{L^I/K} \circ \mathrm{N}_{L/L^I}(x)).$$

□

3.6 The discriminant and the different

In this section, we will describe two effective divisors which detect the ramification of a finite morphism of Dedekind schemes $f : X \to Y$.

Proposition 3.50. *If $f : X \to Y$ is a finite morphism of Dedekind schemes, then f_*O_X is locally free over O_Y. That is, for any $P \in Y$, we may find an open $U \subset Y$ containing P such that $f_*O_X(U)$ is a free $O_Y(U)$-module.*

Proof. Let n be the minimal number of generators needed to generate $f_*O_X(U)$ as an $O_Y(U)$-module, as U ranges over the open sets containing P, and U' an open set with module generators $\{s_i\}_{i=1}^n \in f_*O_X(U')$ witnessing this minimum.

If these were $O_Y(U)$ linearly dependent, then we would have a nonzero relation

$$\sum_{i=1}^{n-1} a_i s_i = a_n s_n.$$

Pick a uniformiser π, restricting U' if necessary to ensure that π has $v_Q(\pi) = 0$ for all $Q \neq P \in U'$.

Multiplying by a suitable (possibly negative) power of π, we may assume one of the a_i has $v_P(a_i) = 0$. But then passing to a subset U'' of U' that contains P, on which that specific a_i is invertible yields a smaller generating set. This is possible since $v_P(a_i) = 0$. Thus, our original generating set must have no relations, giving the desired freeness. □

For an affine subset U on which $f_*O_X(U)$ is free over $O_Y(U)$, let $\{s_i\}_{i=1}^n$ be a basis, and consider the matrix of the trace form with respect to this basis $M = [\operatorname{Tr}(s_i s_j)]_{i,j}$.

Definition 3.51 (The discriminant divisor)**.** We define the discriminant divisor over U to be $\mathrm{div}(\det(M))$. Since the trace form is natural with respect to retrictions, and the localisation of a basis is a basis, we see that viewing these as sections of $\mathrm{Div}(X)$ as a sheaf, we obtain a divisor on all of X. This divisor is the discriminant divisor $\Delta_{X/Y}$.

This well defined, since on U, $\mathrm{div}(\det(M))$ is independent of choice of basis since the determinant of the base change matrix is invertible in $O_Y(U)$.

Remark 3.52. In algebraic number theory, the discriminant is usually defined as an ideal, but we have chosen to define the discriminant at as a divisor for simplicity. We will see in Proposition 4.16 how to relate divisors and ideals on a Dedekind scheme.

Note that we could have defined the discriminant divisor as a sum of

$$v_p(\det(M)) \cdot P$$

over all points of Y. By completing, we can decompose this expression further, since the completion of $(f_*O_X)_P$ at $\mathfrak{p}$ splits orthogonally as in corollary 3.27. We may choose a basis of each free submodule, and note that the determinant is multiplicative in the presence of this direct sum decomposition.

Therefore, the semilocal discriminant $v_P(\det(M)) \cdot P$ can be computed as the sum of the discriminants of each point Q over P, defined by $v_P(\det[\mathrm{Tr}_{Q/P}(c_i c_j)])$ for $\{c_i\}_{i=1}^m$ a basis of $\widehat{O}^{\mathfrak{q}}_{X,Q}$ over $O^{\mathfrak{p}}_{Y,P}$.

Theorem 3.53 (The Discriminant detects ramification on the target)**.** *Given a finite separable morphism of Dedekind schemes $f : X \to Y$, a point P of Y ramifies if and only if its contained in the support of $\Delta_{X/Y}$. If the morphism is inseparable, then every point of Y is ramified.*

Proof. The field extension $K(X)/K(Y)$ splits into $K(X)/L/K(Y)$, with $L/K(Y)$ separable, and $K(X)/L$ purely inseparable. Since f is finite and $O_Y(U)$ is Noetherian, we have that the associated map $v_X(L) \to Y$ is finite, so we may assume our morphism factors as a purely inseparable finite morphism followed by a separable finite morphism. Let us prove the insperable claim first.

By the multiplicativity of $e_{Q/P}$, and the fact that inseparable field extensions remain insperable upon extending further, it suffices to check the second claim in the case of a purely inseparable extension.

Expressing an inseparable extension of iterated degree $p = \mathrm{char}(K(Y))$ extensions, it suffices to check the case of a purely inseparable morphism of degree p.

By completing at Q over a point over P, we may assume we are in the ultra-local situation.

If $e_{Q/P}$ is not equal to p, then the residue extension has degree p by Theorem 3.35. Pick an element x of $\widehat{O}^{\mathfrak{q}}_{X,Q}$ that has coset $\bar{x}$ in the residue field $\widehat{O}_{X,Q}/\mathfrak{q}$ not contained in $O_{Y,P}/\mathfrak{p}$. This x is not contained in $O_{Y,P}$, hence generates the field extension, so x^p is integral over $O_{Y,P}$, and contained in $K(Y)$, hence is in $O_{Y,P}$ by integrality. So the coset of x witnesses that the residue extension is not separable.

So now we may assume our morphism is separable. The semilocal trace form on $(f_*O_X)_P$ descends to an $A/\mathfrak{p}$ linear form on $(f_*O_X)_P/\mathfrak{p}$, and this form will be non degenerate if and only if $v_P(\Delta_{X/Y}) = 0$.

Furthermore, this form will be nondegenerate if and only if every local trace form $\mathrm{Tr}_{Q/P}$ is nondegenerate, so after completing, we may assume that we are in the ultra-local case, of Q/P.

By the naturality of the trace, the reduction of the trace $\overline{\mathrm{Tr}_{Q/P}(x)}$ is given by the trace of the $\kappa(P)$ endomorphism $\mu_{\bar{x}}$ on $(f_*O_X)_P/\mathfrak{p}$.

So if $e_{Q/P} > 1$, a uniformiser π will act as a nilpotent linear operator on this quotient, hence $\overline{\mathrm{Tr}}_{Q/P}(\pi x) = 0$ for all x, so the form is degenerate, and the discriminant is nonzero at P. If the residue extension is inseparable, then $\overline{\mathrm{Tr}}_{Q/P}$ will be identically zero in $\kappa(P)$, so associated determinant will lie in $\mathfrak{p}$.

Finally, if the residue extension is separable, and $e_{Q/P} = 1$, then the residual trace form $\overline{\mathrm{Tr}}_{Q/P}$ equals the field trace form of $\kappa(Q)/\kappa(P)$, and this is nondegenerate, hence the trace form is nondegenerate, so $v_P(\Delta_{X/Y}) = 0$ □

The discriminant is a somewhat coarse indication of ramification, since it is only on the base Y, but we also have a divisor detecting ramification up in X.

Definition 3.54 (The different divisor)**.** We define the divisor $\mathcal{D}_{X/Y}$ as a sum over points in X, as

$$v_Q(\mathcal{D}_{X/Y}) = \max_{0 \neq y \in K(X)} \left\{ v_Q(y) \;\middle|\; \mathrm{Tr}_{Q/P}(y^{-1}x) \in \widehat{O}^{\mathfrak{p}}_{Y,P} \text{ for all } x \text{ in } \widehat{O}^{\mathfrak{q}}_{X,Q} \right\}.$$

That is, the value of the different at a point Q is the highest order of vanishing a function y can have at Q, such that $\mathrm{Tr}_{Q/P}(y^{-1}x) \in \widehat{O}^{\mathfrak{p}}_{Y,P}$ for all $x \in \widehat{O}^{\mathfrak{q}}_{X,Q}$. This can be interpreted as y^{-1} acting like an element of $\widehat{O}^{\mathfrak{q}}_{X,Q}$ with respect to the trace pairing.

Proposition 3.55. *The different $\mathcal{D}_{X/Y}$ is a well defined effective divisor on X.*

Proof. First, that $v_P(\mathcal{D}_{X/Y})$ is finite and positive. Proposition 3.15 gives that this number is positive. Since $\widehat{O}^{\mathfrak{q}}_{X,Q}$ is finitely generated over $\widehat{O}^{\mathfrak{p}}_{Y,P}$, this number is finite, multiplying any basis element by elements of $O^{\mathfrak{p}}_{Y,P}$ and summing them can only make this valuation more negative.

So it remains to show that $v_Q(\mathcal{D}_{X/Y}) = 0$ for all but finitely many Q in X. Pick an affine open set U where $f_*O_X(U)$ is free over $O_Y(U)$. Then the trace form yields an injective $f_*O_X(U)$ linear map

$$f_*O_X(U) \to \hom_{O_Y(U)}(f_*O_X(U), O_Y(U))$$

given by $x \to \mathrm{Tr}(x\_)$. This map is an isomorphism after inverting finitely many elements of $f_*O_X(U)$, and we see directly from the definition that if this map is an isomorphism at Q, we have $v_Q(\mathcal{D}_{X/Y}) = 0$. So the support of $\mathcal{D}_{X/Y}$ is contained in a finite set. □

Remark 3.56. Similar to our definition of the discriminant, for simplicity we have opted to give this as our definition of the different divisor. The reader comfortable with quasicoherent sheaves may find the statement of Proposition 4.61 a more enlightening definition.

The different is a more refined measurement of ramification than the discriminant, the different detects ramification on the source X.

Theorem 3.57. *Let $f : X \to Y$ be a separable finite morphism of Dedekind schemes. A point Q is ramified in X if and only if its contained in the support of the different.*

Proof. Since the different is defined locally, and ramification is local, we may assume the local case. Let $B = \widehat{O}^{\mathfrak{q}}_{X,Q}$, and $A = \widehat{O}^{\mathfrak{p}}_{Y,P}$.

In the unramified case, pick $b \in B^*$ with $\mathrm{Tr}_{Q/P}(b) = 1$, and consider arbitrary $y = \pi^n u$, with $n \geq 0$. If $\mathrm{Tr}_{Q/P}(y^{-1}x) \in A$ for all $x \in B$, then set $x = \pi^{n-1}u^{-1}b$, and we may compute

$$\mathrm{Tr}_{Q/P}(y^{-1}x) = \mathrm{Tr}_{Q/P}(\pi^{-1}b) = \pi^{-1}\,\mathrm{Tr}_{Q/P}(b) = \pi^{-1}.$$

So we have that $n = 0$, and $v_Q(\mathcal{D}_{X/Y}) = 0$.

In the ramified case, if the residue extension is inseparable, then we claim that $\mathrm{Tr}(\pi^{-1}x) \in A$ for all $x \in B$. Since the trace form on residue fields is identically zero, we have $\mathrm{Tr}(\gamma\pi^{-1}x) \in (\gamma)$. Thus by A linearity of the trace, we have $\mathrm{Tr}(x\pi^{-1}) \in A$ for all $x \in B$. So $v_P(\mathcal{D}_{X/Y}) \geq 1$.

In the ramified, separable residue extension case, we claim that $\mathrm{Tr}(\pi^{1-e}x) \in A$ for all $x \in B$. Note that $\pi^{1-e}x\gamma$ acts as a nilpotent linear transformation on $B/\mathfrak{p}$, so has zero trace there, which is equivalent to $\mathrm{Tr}_{Q/P}(\pi^{1-e}x\gamma) \in (\gamma)$, so by A linearity we have $\mathrm{Tr}(\pi^{1-e}x) \in A$, so $v_Q(\mathcal{D}_{X/Y}) \geq e-1$. □

As could be suspected, the disciminant and the different divisors are related, and we have the following theorem.

Theorem 3.58. *The pushforward of the different divisor $\mathcal{D}_{X/Y}$ is the discriminant divisor $\Delta_{X/Y}$.*

$$f_*\mathcal{D}_{X/Y} = \Delta_{X/Y}.$$

We will prove this theorem next chapter after developing some more theory regarding sheaves of modules.

Chapter 4

Sheaves of modules

We will be interested in sheaves of modules over Dedekind schemes in this chapter, so let's start by interpreting R-modules as sheaves of O_R-modules over $\mathrm{Spec}(R)$.

For an R-module M, we may construct a sheaf $\tilde{M}$ of O_R-modules on $(\mathrm{Spec}(R), O_R)$, by sheafifying the presheaf of O_R-modules given by

$$M'(U) := M \otimes_R O_R(U).$$

By a similar argument to the proof of Theorem 2.30, one may show that the canonical map $M \to \tilde{M}(\mathrm{Spec}(R))$ is an isomorphism. This construction is functorial, so we obtain a functor R-Mod to sheaves of O_R-modules over $\mathrm{Spec}(R)$.

Similarly to Theorem 2.32, we have the following result, the proof of which can be found at proposition 2.5.1 in [Har77].

Theorem 4.1. *The functor $M \to \tilde{M}$ is a fully faithful embedding of R-Mod into the category of sheaves of O_R-modules on* $\mathrm{Spec}(R)$.

Definition 4.2 (Affine quasicoherent sheaf)**.** A sheaf of O_R-modules over $\mathrm{Spec}(R)$ is quasicoherent if it is isomorphic to a sheaf in the image of this functor.

Theorem 4.1 states that we have an equivalence between quasicoherent sheaves on an affine scheme $\mathrm{Spec}(R)$, and modules over the ring R. We can intrinsically describe quasicoherent sheaves over an affine scheme as those sheaves which are totally and canonically determined by their global sections. This is to say $\mathcal{F}$ is quasicoherent over an affine scheme $\mathrm{Spec}(R)$ if the canonical map $\widetilde{\mathcal{F}(\mathrm{Spec}(R))} \to \mathcal{F}$ is an isomorphism.

Just like the construction of schemes from affine schemes, we can globalise this to obtain the definition of a quasicoherent sheaf of O_X-modules over an arbitrary scheme X.

Definition 4.3 (Quasicoherent sheaf on a scheme)**.** A sheaf $\mathcal{F}$ of O_X-modules on a scheme X is quasicoherent if for all open affines U in X, we have $\mathcal{F}|_U$ is a quasicoherent sheaf on U.

Remark 4.4. This is equivalent to the seemingly weaker property that $\mathcal{F}|_{U_i}$ is quasicoherent on some affine open cover $\{U_i\}_{i\in I}$ of X, a proof of which can be found as proposition 2.5.4 of [Har77].

From this, we also have the following theorem, analogous to Theorem 2.21 of chapter 2.

Theorem 4.5 (Descent of quasicoherent sheaves)**.** *For any open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of a scheme X, we have an equivalence of categories between quasicoherent sheaves on X and quasicoherent sheaves on our cover $\mathcal{U}$ with descent data.*

$$\mathcal{F} \to \{\mathcal{F}|_{U_i}, \gamma_{i,j}\}_{i,j\in I}$$

From this, if we take $\{U_i\}_{i\in I}$ to be an affine open cover of a scheme X, we see that quasicoherent sheaves can be viewed as a families of modules, along with glueing data.

In view of this, many familiar constructions on modules become operations on quasicoherent sheaves. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of quasicoherent O_X-modules over a scheme X.

Constructions 4.6.

i) The direct sum sheaf $\mathcal{F} \oplus \mathcal{G}$ has sections over U given by $\mathcal{F}(U) \oplus \mathcal{G}(U)$.

ii) The tensor product sheaf $\mathcal{F} \otimes \mathcal{G}$ is given as the sheafification of the presheaf $\widehat{\mathcal{F} \otimes \mathcal{G}}(U) = \mathcal{F}(U) \otimes_{O_X(U)} \mathcal{G}(U)$.

iii) The internal hom-sheaf $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ has sections over U the sheaf homomorphisms $\mathrm{hom}(\mathcal{F}|_U, \mathcal{G}|_U)$, with the natural $O_X(U)$ action. The restriction maps come from restricting a morphism to $\mathrm{hom}(\mathcal{F}|_V, \mathcal{G}|_V)$.

iv) The kernel of a morphism $\mathcal{F} \xrightarrow{f} \mathcal{G}$ of O_X-modules is the subsheaf of $\mathcal{F}$ with section over U given by $\ker(f)(U) = \ker(f|_U)$.

v) The cokernel of a morphism of O_X-modules $\mathcal{F} \xrightarrow{f} \mathcal{G}$ is the sheafification $\mathrm{coker}(f)$ of the cokernel presheaf $\widehat{\mathrm{coker}(f)}(U) = \mathrm{coker}(f|_U)$.

vi) For a morphism $\mathcal{F} \xrightarrow{f} \mathcal{G}$ of sheaves, we have the image presheaf of f in $\mathcal{G}$ given by $\widehat{\mathrm{im}}(\mathcal{F})(U) = \mathrm{im}(f|_U)$. The natural inclusion into $\mathcal{G}$ uniquely factors through its sheafification, and this subsheaf of $\mathcal{G}$ is the image sheaf $\mathrm{im}(f)$.

Definition 4.7. A sheaf $\mathcal{F}$ of O_X-modules is free if $\mathcal{F} \cong \bigoplus_{i \in I} O_X$, and $\mathcal{F}$ is locally free if for some open cover $\{U_i\}_{i \in I}$ of X, each $\mathcal{F}|_{U_i}$ is a free sheaf of $O_X|_{U_i}$-modules.

Remark 4.8. One may note that these operations and properties are precisely the usual ones in R-Mod for quasicoherent sheaves over an affine scheme $\mathrm{Spec}(R)$.

One may note that only sometimes sheafification of the obvious presheaf was needed. This was required for the constructions that do not preserve limits, such as taking tensor products, and taking colimits.

Let X be a scheme that admits an open cover of affines $\{U_i\}_{i \in I}$, with each $U_i \cap U_j$ affine. In view of Theorem 4.5, we may note that these operations respect localisation, so extend naturally to functors on the category of modules along with descent data. So these constructions can alternatively be constructed by applying them on the modules of an affine open cover, then glueing.

From this point, we will identify quasicoherent sheaves over affine schemes with their corresponding modules.

Definition 4.9. A sequence of morphisms of quasicoherent sheaves of O_X-modules $\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{K}$ is exact if for all open affine subschemes the following sequence is exact.

$$\mathcal{F}|_U \xrightarrow{f|_U} \mathcal{G}|_U \xrightarrow{g|_U} \mathcal{K}|_U$$

This is equivalent to $\mathrm{im}(f) = \ker(g)$ as subsheaves of $\mathcal{G}$.

Crucially, the restriction of this sequence is not required to be exact over all open sets. In general the "sections over U" functor $\Gamma(\_, U) : \mathcal{F} \to \mathcal{F}(U)$ will not be right exact. It has been shown in [Gro57] that the right derived functors of $\Gamma(\_, U)$ exist, and furthermore can reasonably be computed, since the exactness over affines allows for computations akin to Čech cohomology.

Since the stalk of $\mathcal{F}$ at P can be computed with reference to an open affine U containing P, we have that $\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{K}$ is exact if and only if for all $P \in X$, the sequence of $O_{X,P}$-modules $\mathcal{F}_P \xrightarrow{f} \mathcal{G}_P \xrightarrow{g} \mathcal{K}_P$ is exact.

Definition 4.10. A morphism of sheaves $\mathcal{F} \xrightarrow{f} \mathcal{G}$ is injective (resp surjective) if its kernel (resp cokerel) is the zero sheaf.

Quasicoherent sheaves over a scheme X inherit some of the excellent category theoretic properties of R-Mod.

Theorem 4.11. *Quasicoherent sheaves on a scheme form a closed symmetric monoidal abelian category with an internal hom functor* Hom. *In particular, constructions 4.6 preserve the quasicoherence property, the canonical map* $\mathrm{coker}(f) \to \mathrm{im}(f)$ *is an isomorphism, and we have a canonical isomorphism of sheaves*

$$\mathrm{Hom}(\mathcal{F} \otimes \mathcal{G}, \mathcal{K}) \cong \mathrm{Hom}(\mathcal{F}, \mathrm{Hom}(\mathcal{G}, \mathcal{K})).$$

Observe that for a quasicoherent sheaf $\mathcal{F}$ on X a scheme, on each open affine U, there exists an exact sequence of O_X-modules ${O_X}^{\oplus I} \to {O_X}^{\oplus J} \to \mathcal{F} \to 0$, given by chosing a presention of the module $\mathcal{F}|_U$.

Remark 4.12. This can also be taken as a definition of quasicoherence, a sheaf $\mathcal{F}$ of O_X-modules is quasicoherent if for every point, we have a neighbourhood U such that $\mathcal{F}|_U$ is the cokernel of a morphism of free $O_X|_U$-modules.

Let's now upgrade the ideal-closed subscheme correspondence of 2.34 to the non-affine situation.

Definition 4.13. A sheaf of ideals $\mathcal{I}$ on a scheme X is a quasicoherent subsheaf of O_X.

On each affine open subscheme $\mathrm{Spec}(R) \cong U \subset \mathrm{Spec}(X)$, we have that $\mathcal{I}|_U$ corresponds to the sheaf of O_R-modules associated to an ideal of R. So a sheaf of ideals can be viewed as a compatible family of ideals in the family of rings that glue to X.

Theorem 4.14. *The map sending each closed subscheme to the kernel of the associated map* $O_X \to f_*O_Z$ *yields a bijection between closed subschemes of* X *and sheaves of ideals of* X*.*

Proof. First, note that f_*O_Z is quasicoherent, as it is locally the sheaf associated to the module $O_X(U)/I$ for some ideal I. Since kernels of quasicoherent sheaves are quasicoherent, we see that this map is well defined, and on each affine yields an ideal subsheaf of O_R. Conversely, given a sheaf of ideals, to obtain the corresponding closed subscheme, we take the closed subset Z of X on which every section of $\mathcal{I}$ vanishes, where defined. The sheaf of rings is given by $O_X/\mathcal{I}(Z \cap U) = O_X/\mathcal{I}(U)$, noting that $O_X/\mathcal{I}(U)$ is canonically isomorphic to $O_X/\mathcal{I}(V)$ if $U \cap Z = V \cap Z$, so this is well defined. This yields locally ringed space, and by checking locally, we have that this is the inclusion of the closed subscheme $\mathrm{Spec}(R/I) \subset \mathrm{Spec}(R)$. □

Returning to the situation of Dedekind schemes, we may now give a canonical bijection between negative divisors on X with two other classes of objects.

Definition 4.15 (Divisor of a closed subscheme)**.** Let X be a Dedekind scheme. To any closed subscheme Z of X, we associate a divisor $\mathrm{div}(Z)$ of X by the following procedure. The ring $O_Z|_{U\cap Z}$ is given as a quotient of $O_X|_U$ by an ideal $\mathcal{I}|_U$ on each affine open set U, and since $\mathcal{I}|_U = \prod_{i=1}^n \mathfrak{p}_i^{e_i}$, we give the divisor $\mathrm{div}(Z)|_U$ on U as $\sum_{i=1}^n -e_i P_i$. These local assignments of data are compatible with restriction, hence glue to the divisor $\mathrm{div}(Z)$.

Intuitively, we are just globalising the affine correspondence of nonzero ideals and finite nonegative sums of prime ideals of Theorem 2.48. It should be noted that we are introducing a sign change compared to the naive bijection between these objects. This will be necessary to avoid unwelcome sign changes later.

Proposition 4.16. *The previous construction gives a bijection between negative divisors on X and closed subschemes of X.*

Proof. Any closed subscheme is determined by its values on each affine open U, so it remains to construct a closed subscheme for any negative divisor D. This is equivalent by Theorem 4.14 to giving the corresponding quasicoherent sheaf of ideals, for which we take

$$L(D)(U) := \bigcap_{P\in U} O_{X,P} \cdot \mathfrak{p}^{-v_P(D)} = \{x \in K(X) | v_P(x) + v_P(D) \geq 0 \text{ for all } P \in U\}$$

This is quasicoherent since on each affine U, it is just the sheaf associated to the ideal $I = \prod_{P\in U} \mathfrak{p}^{-v_P(D)}$. □

So on a Dedekind scheme, we have canonical bijections between the following three objects.

- Negative divisors on X.
- Nonzero sheaves of ideals on X.
- Proper closed subschemes of X.

We will now be considering a more restricted class of sheaves over our Dedekind schemes, a generalisation of finitely presented modules.

Definition 4.17 (Coherent sheaf on a Dedekind scheme)**.** A quasicoherent sheaf $\mathcal{F}$ of O_X-modules over a Dedekind scheme X is coherent if for all affines $U \subset X$, $\mathcal{F}|_U$ is finitely generated as an $O_X|_U$-module. Equivalently, that the local presentation of Remark 4.12 can be taken to be finite.

Remark 4.18. This is not the most general definition of a coherent sheaf, but is equivalent to the general definition found in [Sta19a] for any scheme that is locally Noetherian.

One may show that for a Dedekind scheme X, or more generally a locally Noetherian scheme, that the category of coherent sheaves is closed under all the constructions 4.6, so forms a full abelian subcategory of all quasicoherent sheaves on X.

We will be extensively considering coherent sheaves of an especially nice form, those coherent $\mathcal{F}$ that are locally free as O_X-modules.

It would be intellectually dishonest to not discuss the geometric intepretation of locally free sheaves, even though we will work entirely algebraically. Consider a real vector bundle P over a manifold M. The sheaf of sections of P is a module over the sheaf of continuous real valued functions on M, and one may without difficulty show that this sheaf is locally free. The suitable generalisation of this holds for our schemes, one may show that with the suitable definition of a vector bundle on scheme, we have a bijection between vector bundles on X and locally free sheaves of O_X-modules on a scheme X, see [Sta19b]. We will not use this interpretation explicitly at any point, but we encourage the geometrically minded reader to keep this in mind.

Lemma 4.19. *A coherent sheaf $\mathcal{F}$ on a Dedekind scheme X is locally free if and only if it has each stalk $\mathcal{F}_P$ free over $O_{X,P}$.*

Proof. Since this question is local, we may assume X is affine. So we need to show that a finitely generated module M over a Noetherian ring R is locally free if and only if its localisations M_P are free. Let $\{e_i\}_{i=1}^n$ be a basis of M_P. By clearing potential denominators, we have that e_i are elements of $R[1/f] \otimes_R M$. So we have a map $R[1/f]^n \to R[1/f] \otimes_R M$ which induces an isomorphism when localised at P. Since R is Noetherian, the kernel and cokernel of this map are finitely generated $R[1/f]$-modules that become 0 when localised at P. So there exists $a \in R$ such that a annihilates the kernel and cokernel of this map, so this map is an isomorphism once tensored with $R[1/a]$. So $R[1/f, 1/a] \otimes M$ is free over $R[1/f, 1/a]$, and the claim follows. □

4.1 Functors between sheaf categories

Given a finite morphism of Dedekind schemes $(f, f^\times) : (X, O_X) \to (Y, O_Y)$, we have two primary functors relating the categories of coherent sheaves over each.

Remark 4.20. These functors exist for more general schemes, and for more general maps, but as we will principally be dealing with Dedekind schemes, we will work in this lower level of generality. We will try to indicate where the general theory significantly diverges from this special case.

Our first functor is the pushforward f_*. We may naturally endow the pushforward of a coherent sheaf in the sense of definition 2.6 with an O_Y-module structure.

Definition 4.21. Let $f : X \to Y$ be a finite morphism of Dedekind schemes. The pushforward functor $f_* : \mathrm{Coh}(X) \to \mathrm{Coh}(Y)$ is given on a sheaf $\mathcal{F}$ by

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}U)$$

on open sets, with the restriction maps $f_*\mathcal{F}(U) \to f_*\mathcal{F}(V) = \mathrm{Res}^{f^{-1}U}_{f^{-1}V}$. The O_Y-module structure is given by $x \cdot (v) = f^\times(x)v$.

We now need to check that this sheaf of O_Y-modules is coherent. First, let's note that if $\mathcal{F} \to \mathcal{G} \to \mathcal{K}$ is exact, then so is $f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{K}$ since this sequence is exact on the open cover U_i, since $f^{-1}U_i$ are affine from corollary 3.6.

So by applying the functor f_* to a local presentation of $\mathcal{F}$ on U, it suffices to show that f_*O_X is coherent as a sheaf of O_Y-modules. Since we have an affine open cover $f^{-1}U_i$ of X, it suffices to check the affine case. Let R, S be rings,with M an R-module, $f \in R$ and injective ring morphism $R \to S$. If Res^R_S is the restriction functor R-Mod$\to$ S-Mod, then we have a natural isomorphism

$$\mathrm{Res}^R_S M \otimes_S S[1/f] \cong \mathrm{Res}^R_S(M \otimes_S R[1/f]).$$

This gives an isomorphism $f_*O_R \cong \mathrm{Res}^R_S R \otimes_S O_S$ of presheaves on $\mathrm{Spec}(S)$, which gives that f_*O_R is given on affines U as the sheaf associated to the module $\mathrm{Res}^R_S O_R(U)$, which is finitely generated since our map is finite.

Note that on affines, and on stalks, this functor is given by restriction of modules.

Remark 4.22. This functor is defined for most morphisms of schemes, but only on the level of quasicoherent sheaves. The fact that our map has a cover of affine open sets mapping to affines is just a convienience we used in the proof, but the fact that it preserves coherence relies critically on the finiteness of our map.

Our second functor f^* is the pullback, analogous to our inverse image functor of definition 2.8. Recall from corollary 3.38 that finite morphisms of Dedekind schemes are open maps.

Definition 4.23. Given a finite morphism of Dedekind schemes $f : X \to Y$, we define the pullback $f^*\mathcal{F}$ to be the coherent sheaf of O_X-modules given by the sheafification of the presheaf

$$\tilde{f}^*\mathcal{F}(U) := O_X(U) \otimes_{O_Y(f(U))} \mathcal{F}(f(U))$$

Here O_X acts on the left factor, and the $O_Y(f(U))$ algebra structure on $O_X(U)$ is given by the composition

$$O_Y(f(U)) \to f_*O_X(f(U)) = O_X(f^{-1}(f(U))) \xrightarrow{Res} O_X(U).$$

Let's check that this is a coherent sheaf. First, note directly from the definition that $f^*O_Y \cong O_X$. This functor is locally given by taking the tensor product on affines, then sheafifying. Since the tensor product is right exact on modules, it preserves local presentations, so preserves coherence.

Note that on affine open sets of the form $f^{-1}(U)$ in X for U open affine in Y, this map is given by taking the tensor product as modules, and is also given by this on stalks.

Remark 4.24. In general, for maps that arent open, we need to take a colimit similarly to the definition of the inverse image sheaf.

Theorem 4.25. *For a finite map $f : X \to Y$ of Dedekind schemes, we have an adjunction*

$$\mathrm{Coh}(Y) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Coh}(X) \ .$$

Furthermore, the canonical morphism defines an isomorphism of sheaves

$$f_* \operatorname{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong \operatorname{Hom}(\mathcal{F}, f_*\mathcal{G}).$$

Proof. Note that for a coherent sheaf $\mathcal{F}$, we have that $f^*f_*\mathcal{F}$ is the sheafification of the presheaf $O_X(U) \otimes_{O_Y(f(U))} \mathcal{F}(f^{-1}f(U))$, and $f_*f^*\mathcal{G}$ is the sheafification of the presheaf $O_X(f^{-1}(U)) \otimes_{O_Y(U)} \mathcal{G}(U)$.

The unit of this adjunction is given by the composition of the natural map of presheaves

$$\mathcal{G}(U) \to O_X(f^{-1}(U)) \otimes_{O_Y(U)} \mathcal{G}(U)$$

$$s \to 1 \otimes s$$

and the sheafification map to $f_* f^* \mathcal{F}$.

The co-unit of this adjunction is given by the unique sheafification-induced map associated to the presheaf morphism

$$O_X(U) \otimes_{O_Y(f(U)} \mathcal{G}(f^{-1}f(U)) \to \mathcal{G}(U)$$

$$a \otimes s \to a \cdot \mathrm{Res}_U^{f^{-1}fU}(s).$$

To check the triangle identities, we can check that the composities are equalities on stalks. So let $B = O_{X,Q}$, $A = O_{Y,f(Q)}$. we see that $f^*\mathcal{F} \to f^* f_* f^* \mathcal{F} \to f^* \mathcal{F}$ is given on the stalk at Q as as

$$B \otimes_A \mathcal{F}_Q \to B \otimes_A B \otimes_A \mathcal{F}_Q \to B \otimes_A \mathcal{F}_Q.$$

$$a \otimes s \to 1 \otimes a \otimes s \to 1 \cdot a \otimes s = a \otimes s$$

For the other triangle identity, let $C = (f_* O_X)_P$. Then $f_*\mathcal{G} \to f_* f^* f_* \mathcal{G} \to f_* \mathcal{G}$ is given by

$$f_*\mathcal{G}_P \to C \otimes_A f_*\mathcal{G}_P \to f_*\mathcal{G}_P.$$

$$s \to 1 \otimes s \to 1 \cdot s = s$$

The second assertion follows by noting the adjunction preserves the O_Y-module structure that can naturally be given on both sides of $\hom(f^*\mathcal{F}, \mathcal{G}) \cong \hom(\mathcal{F}, f_*\mathcal{G})$. □

These two functors exist in almost complete generality for morphisms of schemes, but now we will consider a third functor, much more specific to our situation. This functor $f^!$ is a right adjoint to f_*.

Since this is more involved to construct than the previous functors, we will first define it on affines, then glue using Theorem 4.5.

Definition 4.26. Given a finite morphism $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of affine Dedekind schemes, and M a finitely generated A-module, we define

$$f_B^! M := \hom_A(\mathrm{Res}_A^B B, M).$$

That is, $f_B^! M$ is the A-module morphisms from B to M, with B action given by multiplication on B.

Note that since B is finite over A, $f^!_B M$ is also finitely generated.

This construction yields a functor from finitely generated A-modules to finitely generated B-modules.

Theorem 4.27. *For a finite morphism $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of affine Dedekind schemes, we have an adjunction*

$$\mathrm{Coh}(\mathrm{Spec}(B)) \underset{f^!_B}{\overset{f_*}{\rightleftarrows}} \mathrm{Coh}(\mathrm{Spec}(A)).$$

Proof. We define the unit as the map $N \to f^! f_* N$ that takes an element m of M to the B linear map $1 \xrightarrow{c_m} m$, and intepretes this as an A linear morphism $\mathrm{Res}^B_A B \to M$. For the co-unit, any A linear morphism $\mathrm{Res}^B_A B \to N$ maps 1 to an element of N, giving a natural evaluation map $f_* f^! N \to N$. To verify the triangle identities, we just need to note that the following compositions are both the identity.

$$m \to c_m \to c_m(1) = m$$

$$\phi \to c_\phi \to c_\phi(1) = \phi.$$

□

We aim now to construct the functor $f^!$ for non affine Dedekind schemes, by glueing these locally defined $f^!$. To do this, we will need this construction to behave well under localisation.

Lemma 4.28. *Let $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be a finite morphism of affine Dedekind schemes, and let U be an open affine in $\mathrm{Spec}(A)$. Let*

$$V = f^{-1}(U),\ B' = O_B(V),\ A' = O_A(U).$$

Then for any finitely generated A-module M, we have a canonical isomorphism

$$B' \otimes_B f^!_B(M) \xrightarrow{\gamma^B_{B'}} f^!_{B'}(A' \otimes_A M).$$

This isomorphism is compatible with restriction, if we have affine inclusions $U'' \subset U' \subset U$, with corressponding ring $B \subset B' \subset B''$, we have $\gamma^{B'}_{B''} \circ \gamma^B_{B'} = \gamma^B_{B''}$.

Proof. We have the natural isomorphism

$$B' \otimes_B \hom_A(B, M) \cong \hom_A(B', M)$$

given by $(1 \otimes \phi)(b') = b'\phi(1)$, extending by B' linearity. Composing, we obtain a morphism

$$\hom_A(B', M) \to \hom_{A'}(B', A' \otimes_A M).$$

This map is injective, if ϕ is in the kernel, and $\phi(s) = m$, then m is annihilated by some $f_s \in A$ which becomes invertible in A'. By finite generation, we may choose an $F \in A$ which annihilates everything in the image of ϕ. But F acts invertibly on B', so since $\phi(Fs) = 0$ for all $s \in B'$, we have $\phi = 0$.

Now let's check that this is surjective. Given $\gamma : B' \to A' \otimes_A M$, then picking a generating set $\{s_i\}_{i=1}^n$ of B', we have $\gamma(s_i) = \frac{m_i}{f_i}$, so letting $F = \prod_{i=1}^n f_i$, we have $F\gamma$ has image contained in M, so $\frac{1}{F} \cdot F\gamma$ shows that the map is surjective, since $\frac{1}{F} \in B'$.

For the compatibility, note that $\gamma_{B'}^B(1 \otimes \phi) = i_{B'}^B \circ \phi$, where $i_{B'}^B : B \to B'$ is the inclusion. □

At this point, we claim that for any Dedekind scheme X, there exists an affine open cover $\{U_i\}_{i\in I}$ of Y such that each intersection $U_i \cap U_j$ is also affine. This holds since our schemes are seperated in the sense of [Har77], and while this is implied by our definition 2.54 of seperatedness, the author could not find a proof of this covering fact using the material solely in this thesis.

We will now use such a cover $\{U_i\}_{i\in I}$ of Y to construct the functor $f^!$ for a finite morphism of Dedekind schemes $f : X \to Y$.

Let $V_i := f^{-1}(U_i)$ be the corresponding affine cover of X. Define $B_i := O_X(V_i)$, noting that it is the integral closure of $A_i := O_Y(U_i)$ in $K(X)$. Also let

$$B_{i,j} := O_X(V_i \cap V_j),\ A_{i,j} := O_Y(U_i \cap U_j).$$

Given a coherent sheaf $\mathcal{F}$ on Y, we express it as its family of modules with descent data $\mathcal{F} = (M_i, A_{i,j} \otimes_{A_i} M_i \xrightarrow{\nu_{i,j}} A_{i,j} \otimes_{A_j} M_j)$.

Applying the functor $f^!_{B_{i,j}}$ and composing with the natural isomorphisms of Lemma 4.28, we obtain a sheaf on X given by the glueing of the following data

$$(f^!_{B_i} M_i, B_{i,j} \otimes_{B_i} f^!_{B_i} M_i \xrightarrow{\gamma^{-1}\nu_{i,j}\gamma} B_{i,j} \otimes_{B_j} f^!_{B_j} M_j)$$

These isomorphisms are valid descent data by the compatibility part of 4.28.

We aim to define the unit and co-unit of the adjunction in terms of these presentations of our sheaves as modules and descent data. We may apply the unit and co-unit of Theorem 4.27 on the modules, so it remains to check that these respect the natural descent data on both sides.

So if our sheaves are

$$\mathcal{F} = (M_i, A_{i,j} \otimes_{A_i} M_i \xrightarrow{\nu_{i,j}} A_{i,j} \otimes_{A_j} M_j)$$

on Y and

$$\mathcal{G} = (N_i, B_{i,j} \otimes_{B_i} N_i \xrightarrow{\mu_{i,j}} B_{i,j} \otimes_{B_j} N_j)$$

on X, then we would need to check the commutativity of the following two diagrams.

$$\begin{array}{ccc} B_{i,j} \otimes_{B_i} N_i & \xrightarrow{\mu_{i,j}} & B_{i,j} \otimes_{B_j} N_j \\ \downarrow{\scriptstyle 1\otimes\eta} & & \downarrow{\scriptstyle 1\otimes\eta} \\ B_{i,j} \otimes_{B_i} f^!_{B_i} \operatorname{Res}^{B_i}_{A_i} N_i & \longrightarrow & B_{i,j} \otimes_{B_j} f^!_{B_j} \operatorname{Res}^{B_j}_{A_j} N_j \end{array}$$

$$\begin{array}{ccc} A_{i,j} \otimes_{A_i} \operatorname{Res}^{B_i}_{A_i} f^!_{B_i} M_i & \longrightarrow & A_{i,j} \otimes_{A_j} \operatorname{Res}^{B_j}_{A_j} f^!_{B_j} M_j \\ \downarrow{\scriptstyle 1\otimes\epsilon} & & \downarrow{\scriptstyle 1\otimes\epsilon} \\ A_{i,j} \otimes_{A_i} M_i & \xrightarrow{\nu_{i,j}} & A_{i,j} \otimes_{A_j} M_j \end{array}$$

Where the unlabelled maps are the compositions of the maps of descent data of the functors. This verification is a long simple unfolding of definitions, which we therefore omit.

Noting that the adjunction triangles are verifiable on the level of affines, we see that this functor is a right adjoint to f_*. Since adjoints are unique up to unique isomorphism, we see that this functor is independent of the cover we chose, and we have our desired adjunction.

Theorem 4.29. *For a finite morphism of Dedekind schemes $f : X \to Y$, the pushforward f_* on coherent sheaves admits a right adjoint $f^!$.*

$$\operatorname{Coh}(X) \underset{f^!}{\overset{f_*}{\rightleftarrows}} \operatorname{Coh}(Y)$$

Remark 4.30. This right adjoint $f^!$ of f_* exists in much greater generality than finite morphisms of schemes, after suitably enlarging our category $\operatorname{Coh}(X)$ and $Coh(Y)$. To the author's knowledge however, even though this functor exists in much greater generality, a general concrete description of it is not known, see [Har66].

4.2 The class group and the picard group

Recalling definition 3.42, the class group $\mathrm{Cl}(X)$ of X is given as the quotient of $\mathrm{Div}(X)$ by the principal divisors of the form $\mathrm{div}(f)$. In this section we will define another group based on locally free sheaves, which for Dedekind schemes will be canonically isomorphic to $\mathrm{Cl}(X)$.

Definition 4.31 (Invertible sheaf). A sheaf $\mathcal{F}$ of O_X-modules on a scheme X is invertible if there exists another sheaf $\mathcal{G}$ and isomorphisms $\mathcal{F} \otimes \mathcal{G} \cong O_X \cong \mathcal{G} \otimes \mathcal{F}$.

Invertible sheaves on Dedekind schemes also admit a more concrete description.

Lemma 4.32. *For a sheaf $\mathcal{F}$ of O_X modules on a Dedekind scheme X, the following are equivalent.*

i) $\mathcal{F}$ is invertible.

ii) $\mathcal{F}$ is locally free of rank 1.

Proof. i) implies ii) We follow the proof of this result given at [Sta19c].

Given an isomorphism $\gamma : O_X \to \mathcal{F} \otimes \mathcal{G}$, on U we have locally that $\gamma(1_U) = \sum_{i=1}^{n} a_i \otimes b_i$ for some $a_i \in \mathcal{F}(U)$, $b_i \in \mathcal{G}(U)$. So consider the composite map

$$\mathcal{F}|_U \to \mathcal{F}|_U \otimes \mathcal{F}|_U \otimes \mathcal{G}|_U \to \mathcal{F}|_U$$

given by

$$s \to \sum_i s \otimes a_i \otimes b_i \to \sum_i \gamma^{-1}(s \otimes a_i) b_i.$$

This is an automorphism of $\mathcal{F}|_U$, and factors as $\mathcal{F}|_U \to O_X|_U^n \to \mathcal{F}|_U$, so $\mathcal{F}$ is a direct summand of a coherent sheaf, hence is coherent. Since its stalks are finite free modules, by Lemma 4.19 $\mathcal{F}$ is seen to be locally free.

ii) implies i)

If $\mathcal{F}$ is locally free of rank 1, then the sheaf $\mathrm{Hom}(\mathcal{F}, O_X)$ is an inverse of $\mathcal{F}$. To see this, note the natural evaluation maps

$$\mathcal{F} \otimes_{O_X} \mathrm{Hom}(\mathcal{F}, O_X) \to O_X$$

are isomorphisms on stalks, since stalkwise these are simply the pairing of a free rank one module with its dual. □

Definition 4.33 (The Picard group). The picard group of a scheme X is the abelian group $\mathrm{Pic}(X)$ of isomorphism classes of invertible sheaves on X.

We will now begin relating divisors to invertible sheaves.

Definition 4.34 (Divisor associated to a subsheaf of $K(X)$)**.** Let X be a Dedekind scheme. To any invertible subsheaf L of $K(X)$, we may associate a divisor $\mathrm{div}(L)$ as follows. The stalk L_P is a $O_{X,P}$ submodule of $K(X)_P = K(X)$, and is therefore of the form $\pi^{a_P} O_{X,P}$ for a uniformiser π of $O_{X,P}$, for some $a_P \in \mathbb{Z}$. We define then define $\mathrm{div}(L)$ as

$$\mathrm{div}(L) = \sum_{P \in X} -a_P \cdot P.$$

Remark 4.35. This is a well defined divisor. Any invertible subsheaf of $K(X)$ is free over an open set U, so we have an $x \in K(X)$ such that $L_P = (x) \subset O_{X,P}$ for all $P \in U$. So away from the finite set $\mathrm{div}(x)$, we have that $L_P = O_{X,P}$. Thus, the set of points where $a_P \neq 0$ is finite.

Definition 4.36 (Subsheaf of $K(X)$ associated to a divisor)**.** For any divisor $D = \sum_{P \in X} a_P \cdot P$, we have an invertible subsheaf $L(D)$ of $K(X)$ given by

$$L(D)(U) = \bigcap_{P \in U} \mathfrak{p}^{-a_P} O_{X,P} = \{x \in K(X) | v_P(x) + v_P(D) \geq 0 \text{ for all } P \in U\}.$$

This is invertible since the $K(X)$ multiplication map $L(-D) \otimes_{O_X} L(D) \to O_X$ shows that $L(-D)$ is an inverse.

Note that the operations are mutual inverses, and we obtain a bijection between divisors and invertible subsheaves of $K(X)$.

We have partial orders, group laws, and notions of equivalence on both of these sets, and these are all preserved under this identification.

Proposition 4.37. *For two divisors D, D' on X, we have:*

i) $D \leq D'$ if and only if $L(D) \subset L(D')$ as subsheaves of $K(X)$.

ii) $L(D) \otimes_{O_X} L(D') \cong L(D + D')$

iii) $L(D) \cong L(D')$ if and only if $D = D' + \mathrm{div}(f)$ for some $f \in K(X)$.

Proof. The first proposition is immediate from the definitions, and for *ii*), we have a natural injective morphism $L(D) \otimes L(D') \to K(X)$ given by restricting the multiplication map

$$K(X) \otimes K(X) \to K(X).$$

To identify its image, we may pass to stalks and observe that the corresponding divisors add.

For *iii*), given an isomorphism $\phi : L(D) \to L(D')$, we may tensor it with the constant $K(X)$ sheaf to obtain the following diagram, where all maps are isomorphisms.

$$\begin{array}{ccc} L(D) \otimes K(X) & \longrightarrow & K(X) \\ \big\downarrow{\scriptstyle \phi\otimes\mathrm{id}} & & \\ L(D') \otimes K(X) & \longrightarrow & K(X) \end{array}$$

So ϕ arises as the restriction of an isomorphism $K(X) \to K(X)$, which is multiplication by a nonzero element f of $K(X)$, giving the result upon observing the effect on the corresponding divisors. □

Lemma 4.38 (Invertible sheaves embed in the constant sheaf)**.** *Every invertible sheaf L on X is isomorphic to a subsheaf of $K(X)$.*

Proof. Since L is invertible, it is locally free of rank 1, so its stalk $L_{x_{gen}}$ over the generic point x_{gen} is a one dimensional vector space over $K(X)$. So the defining morphisms $L(U) \to L_{gen}$ yield an injection of L into a constant L_{gen} sheaf. Composing this map with any isomorphism $L_{gen} \cong K(X)$ gives the desired embedding. □

Remark 4.39. This process is not canonical, we arbitrarily chose an isomorphism $L_{gen} \cong K(X)$, and indeed we should not think of abstract invertible sheaves as being subsheaves of $K(X)$.

Corollary 4.40 (The Picard group is the class group)**.** *Embedding an invertible sheaf into $K(X)$ then taking its associated divisor in the class group* $\mathrm{Cl}(X)$ *yields a well defined isomorphism of groups*

$$\mathrm{Pic}(X) \to \mathrm{Cl}(X).$$

Proof. This map is well defined by Lemma 4.38 and is a bijective group homomorphism by Proposition 4.37. □

Remark 4.41. For this proof, all we needed was a good correspondence between divisors and invertible subsheaves of $K(X)$. Taking divisors to be sums of codimension one closed subschemes of X, this correspondence can hold more generally for higher dimensional schemes. For sufficiently nice schemes X, this same isomorphism holds, with an analagous proof. To see this result in a higher dimensional setting, see proposition 2.6.15 of [Har77].

4.3 Modules over a Dedekind scheme

Definition 4.42 (Grothendieck group of an abelian category)**.** Given an abelian category C, we define the Groethendieck group of C, denoted $K_0(C)$, to be the free abelian group on the objects of C, modulo the relations $X+Y-Z$ whenever we have a short exact sequence

$$0 \to X \to Z \to Y \to 0.$$

By equating the positive and negative parts of such a formal expression, each element of $K_0(X)$ has a representative of the form $[X]-[Y]$ for objects X, Y in C.

Remark 4.43. If C is monoidal, then $K_0(C)$ naturally carries the structure of a ring, but we won't be utilising this ring structure.

We may use this construction to give another interpretation of divisors on our Dedekind scheme X, as the Grothendieck group of the category of coherent torsion sheaves.

Definition 4.44. A coherent sheaf T on X is torsion if for an affine open cover U_i, $T|_{U_i}$ is a torsion module over $O_X(U_i)$. Coherent torsion sheaves on X form a full abelian subcategory of $\mathrm{Coh}(X)$, which we denote by$\mathrm{Coh}_{\mathrm{Tor}}(X)$.

Definition 4.45 (Torsion sheaf associated to a negative divisor)**.** For a point P in X, and a positive integer $n \in \mathbb{N}$, we have the associated map of sheaves $L(-nP) \to O_X$, and the cokernel of this map is a coherent torsion sheaf denoted $T(-nP)$. Explicitly, we have

$$T(-nP)(U) = O_{X,P}/\mathfrak{p}^n$$

if $P \in U$, and 0 else.
We define $T(-D)$ for any effective divisor D on X as the cokernel of the inclusion $L(-D) \to O_X$.

These $T(-D)$ give us examples of coherent torsion sheaves on X, and after taking direct sums, yield the whole category $\mathrm{Coh}_{\mathrm{Tor}}(X)$.

Proposition 4.46. *A coherent torsion sheaf T on X a Dedekind scheme can be expressed uniquely as a direct sum*

$$T = \bigoplus_{P\in X} \bigoplus_{i} T(-nP)^{e_{i,P}}.$$

Where $e_{i,P} \in \mathbb{N}$, and are zero for all but finitely many pairs (i, P).

Proof. There exists an affine open U, and a nonzero $f \in O_X(U)$ with $f.s = 0$ for all $s \in T(U)$. Passing to an open U' away from the zeros and poles of f, we see that $T(V) = 0$ if $V \subset U'$. Let $P_1, ...P_n$ be the points contained in $X \setminus U'$, and pick open sets U_i that contain P_i, but not P_j for $j \neq i$, and set $U_0 = U'$.

These $\{U_i\}_{i=0}^n$ are an open cover of X, so we have the following equaliser

$$T(V) \to \bigoplus_{i=0}^{n} T(V \cap U_i) \rightrightarrows \bigoplus_{i,j=0}^{n} T(V \cap U_i \cap U_j).$$

Since $T(V \cap U_i \cap U_j) = 0$ unless $i = j$, we see that

$$T(V) \cong \bigoplus_{i=0}^{n} T(V \cap U_s).$$

If V contained a single P_i only, we see that $T(V) \cong T(V \cap U_i)$, so we see for U'' containing P_i only, $T(U'')$ is isomorphic to the stalk T_{P_i} of T at P. Since each $O_{X,P}$ is a principal ideal domain, we have a decomposition

$$T_P \cong \bigoplus_{n=1}^{m} (O_{X,P}/\mathfrak{p}^n)^{e_n}$$

So $T(V)$ is isomorphic to the direct sum of the stalks of T at the P_i contained in V, giving the desired decomposition. □

Remark 4.47. Note that from this description, we see that every coherent torsion sheaf admits a surjection from O_X^n, since each $T(-nP)$ does.

Corollary 4.48. *The mapping $T(-nP) \to nP$ extends to an isomorphism*

$$K_0(\mathrm{Coh}_{\mathrm{Tor}}(X)) \cong \mathrm{Div}(X).$$

Proof. Note that we have a short exact sequence of $O_{X,P}$-modules

$$0 \to \mathfrak{p}^k/\mathfrak{p}^{k+1} \to O_{X,P}/\mathfrak{p}^{k+1} \to O_{X,P}/\mathfrak{p}^k \to 0.$$

Identifying $\mathfrak{p}^k/\mathfrak{p}^{k+1}$ with $O_{X,P}/\mathfrak{p}$ under multiplication by π^k for π a uniformiser, we obtain the short exact sequence of torsion sheaves

$$0 \to T(-P) \to T(-nP) \to T(-(n-1)P) \to 0.$$

Since both the Grothendieck group and the divisor group are direct sums over the points in X by Proposition 4.46, the result follows. □

Let us now consider the category of all coherent sheaves over X a Dedekind scheme, not just the torsion sheaves.

Theorem 4.49 (Identifying $K_0(\mathrm{Coh}(X))$)**.** *For X a Dedekind scheme, we have a canonical isomorphism*

$$K_0(\mathrm{Coh}(X)) \xrightarrow{\mathrm{ch}} \mathbb{Z} \oplus \mathrm{Pic}(X)$$

as abelian groups, given on invertible sheaves by $L \to (1, [L])$.

We will prove this theorem in a series of steps. First, we will show that arbitrary coherent sheaves aren't too far from locally free sheaves.

Theorem 4.50. *If $\mathcal{F}$ is a coherent sheaf on a Dedekind scheme X, then there exists an invertible sheaf L such that $\mathcal{F} \otimes L$ admits a surjective map from $(O_X)^n$.*

Proof. We will prove this by first showing that any section $s \in \mathcal{F}(U)$ can be extended to a global section $\tilde{s}$ of $\mathcal{F} \otimes L$ for some invertible sheaf L with $L|_U \cong O_X|_U$, such that $\tilde{s}|_U$ is the image of s under the isomorphism $\mathcal{F}|_U \cong \mathcal{F}|_U \otimes L|_U$.

We first consider the torsion subsheaf of $\mathcal{F}$, the subsheaf of sections that have nonzero annihilator. This is the subsheaf $T_{\mathcal{F}}$ given by:

$$T_{\mathcal{F}}(U) = \{s \in \mathcal{F}(U) | \text{ there exists a nonzero } f \in O_X(U) \text{ such that } f.s = 0\}.$$

This sheaf is precisely the kernel of the map of quasicoherent sheaves $\mathcal{F} \to \mathcal{F} \otimes K(X)$. Therefore it is a quasicoherent subsheaf of a coherent sheaf, hence a coherent sheaf since our schemes are Noetherian.

It is also torsion, so by 4.47, admits a surjection from O_X^n for some finite $n \in \mathbb{N}$.

Now if $s \in \mathcal{F}(U)$ is not annihilated by any $f \in O_X(U)$, we can without loss of generality assume U is affine, and we will construct an invertible sheaf L as follows:

Let s^e be the image of s in the stalk over the generic point, $\mathcal{F}_{gen}$. Pick an open affine cover U_i of X, and on each U_i pick a $t_i \in \mathcal{F}(U_i)$, $d_i \in O_X(U_i)$ such that in $\mathcal{F}_{gen}$ we have

$$t_i = d_i s^e.$$

We may take $U_1 = U$, and $d_1 = 1$. We see therefore that the section $d_j t_i - d_i t_j \in \mathcal{F}(U_i \cap U_j)$ is annihilated by some $c_{i,j} \in O_X(U_i \cap U_j)$. From the sheaf property, we may assume that each of these $c_{i,j}$ are all nonzero. Pick a divisor D_s such that D_s is greater than each divisor of the form $\mathrm{div}(c_{i,j} d_j d_i)$.

Now consider on each U_i the sections of $L(D_s) \otimes \mathcal{F}$ given by

$$\frac{1}{d_i} \otimes t_i.$$

On the intersections $U_i \cap U_j$, we have

$$\frac{1}{d_i} \otimes t_i = \frac{c_{i,j} d_j}{c_{i,j} d_j d_i} \otimes t_i = \frac{1}{c_{i,j} d_j d_i} \otimes c_{i,j} d_j t_i = \frac{1}{c_{i,j} d_j d_i} \otimes c_{i,j} d_i t_j = \frac{c_{i,j} d_i}{c_{i,j} d_j d_i} \otimes t_j = \frac{1}{d_j} \otimes t_j.$$

These sections therefore glue to a global section $\tilde{s}$ of $L(D_s) \otimes \mathcal{F}$. We see that $\tilde{s}|_U = s$, giving the desired extension.

Picking a finite affine cover U_i of X, for each i we can pick $s_{1,i}, ..s_{n,i}$ that together generate the cokernel of $T_{\mathcal{F}} \to \mathcal{F}$ over U_i. Taking a divisor $D_{s_{j,i}} \leq D$ for all i, j, and summing the maps $O_X \to L(D) \otimes \mathcal{F}$ associated to the global section $s_{i,j}$, we obtain a morphism

$$O_X^N \to L(D) \otimes \mathcal{F}.$$

By construction, this generates the image of $\mathcal{F}$ in $\mathcal{F} \otimes K(X)$. So now picking a surjection from O_X^M onto the torsion subsheaf of $L(D) \otimes \mathcal{F}$, we may sum these to obtain the desired surjection onto $L(D) \otimes \mathcal{F}$. □

Corollary 4.51. *Every coherent sheaf $\mathcal{F}$ on a Dedekind scheme X has a two step resolution by locally free sheaves. That is, for any coherent $\mathcal{F}$ on X, there exists locally free V_1 and V_2, and a short exact sequence*

$$0 \to V_1 \to V_2 \to \mathcal{F} \to 0.$$

Proof. Tensor the surjection $O_X^n \to \mathcal{F} \otimes L$ with L^{-1} an inverse of L to obtain a surjection from a locally free sheaf $(L^{-1})^n$ to $\mathcal{F}$, and note that the kernel is a coherent subsheaf of a locally free sheaf, so its stalks are free of finite rank, hence this kernel is locally free by Lemma 4.19. □

Definition 4.52. The rank of a coherent sheaf $\mathcal{F}$ over X is the dimension of $\mathcal{F}_{gen}$ as a vector space over $K(X)$, denoted $\mathrm{rk}(\mathcal{F})$.

Observe that the rank of $\mathcal{F}$ is zero if and only if $\mathcal{F}$ is torsion. The map $\mathcal{F} \to \mathrm{rk}(\mathcal{F})$ descends to a map from $K_0(X)$ to $\mathbb{Z}$, since taking the stalk at x_{gen} is exact, and vector space dimension is additive in short exact sequences.

This will be the $\mathbb{Z}$ valued component of the map ch, and now we will define the $\mathrm{Pic}(X)$ valued component.

Definition 4.53 (Determinant of a locally free sheaf)**.** Given a locally free sheaf $\mathcal{F}$ on X, the determinant $\det(\mathcal{F})$ of $\mathcal{F}$ is the invertible sheaf obtained by taking the top exterior power "locally" on $\mathcal{F}$. That is, take an open affine cover U_i on which $\mathcal{F}$ is free, and take the top exterior power on each, and glue using the compatibility of exterior powers and localisation.

For a locally free sheaf $\mathcal{F}$, we define the dual locally free sheaf $\mathcal{F}^*$ to be $\mathrm{Hom}(\mathcal{F}, O_X)$, and we have the following compatibility result.

Proposition 4.54. *As invertible sheaves on X, we have a canonical isomorphism* $\det(\mathrm{Hom}(\mathcal{F}, O_X)) \cong \det(\mathcal{F})^{-1}$

Proof. We have the canonical pairing of top wedge powers of a free module, and globalising this yields the desired natural isomorphism

$$\det(\mathcal{F}) \otimes \det(\mathrm{Hom}(\mathcal{F}, O_X)) \to O_X.$$

□

Lemma 4.55 (Determinant of a short exact sequence)**.** *For a short exact sequence of locally free sheaves,* $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ *we have a canonical isomorphism of invertible sheaves.*

$$\det(\mathcal{F}_1) \otimes \det(\mathcal{F}_3) \cong \det(\mathcal{F}_2)$$

Proof. For a short exact sequence of free R-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

we have the canonical isomorphism $\det(M_1) \otimes \det(M_3) \to \det(M_2)$, on pure alternating tensors as $(v_1 \wedge v_2 \wedge ... \wedge v_n) \otimes (w_1 \wedge w_2 \wedge ... \wedge w_m) \to v_1 \wedge .. \wedge v_n \wedge w_1 \wedge .. \wedge w_m$. This is natural, and compatible with localisation, hence glues to an isomorphism $\det(\mathcal{F}_1) \otimes \det(\mathcal{F}_3) \cong \det(\mathcal{F}_2)$ of locally free sheaves. □

Definition 4.56 (Determinant of a coherent sheaf)**.** Let $0 \to \mathcal{G}_2 \to \mathcal{G}_1 \to \mathcal{F} \to 0$ be a two step locally free resolution of $\mathcal{F}$, a coherent sheaf on a Dedekind scheme X. The determinant $\det(\mathcal{F})$ of $\mathcal{F}$ is defined to be the isomorphism class $[\det(\mathcal{G}_1)] - [\det(\mathcal{G}_2)]$ of $\det(\mathcal{G}_1) \otimes \det(\mathcal{G}_2)^{-1}$ in the Picard group $\mathrm{Pic}(X)$.

Lemma 4.57. *This isomorphism class is independent of the resolution chosen.*

Proof. First, note that the category of locally free sheaves on X is closed under direct sums and kernels, so is closed under taking pullbacks. Consider then the following diagram, where each $P_{i,j}$ is defined to be the appropriate pullback.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_{2,2} & \longrightarrow & P_{1,2} & \longrightarrow & V_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_{1,2} & \longrightarrow & P_{1,1} & \longrightarrow & V_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G}_2 & \longrightarrow & \mathcal{G}_1 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Each row and column of this diagram is a short exact sequence, as can be verified by a simple diagram chase.

From this, we see that

$$\begin{aligned}
[\det(V_1)] - [\det(V_2)] =& [\det(P_{1,1})] - [\det(P_{1,2})] + [\det(P_{2,2})] - [\det(P_{1,2})] \\
=& [\det(\mathcal{G}_1)] - [\det(\mathcal{G}_2)].
\end{aligned}$$

So the class in $\mathrm{Pic}(X)$ is independent of the choice of resolution. □

Thus we obtain a map $\mathcal{F} \to [\det(\mathcal{F})]$ from coherent sheaves to $\mathrm{Pic}(X)$. We claim that this map factors through $K_0(X)$.

Given a short exact sequence of coherent sheaves $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$, and surjection $V \to \mathcal{F}_2$ from a locally free sheaf V, we obtain the following diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_3 & \xrightarrow{\sim} & P_1 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_2 & \longrightarrow & V & \longrightarrow & \mathcal{F}_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \mathcal{F}_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Where once more, the P_i are pullbacks, and each row and column is a short exact sequence. From this, we see that $[\det(\mathcal{F}_2)] = [\det(\mathcal{F}_1)] + [\det \mathcal{F}_3]$, since both sides are the class of $[\det(V)] - [\det(P_1)]$.

Thus we now have a well defined additive map $K_0(X) \to \operatorname{Pic}(X)$.

Note that for an effective divisor D, taking the determinant of the short exact sequence

$$0 \to L(-D) \to O_X \to T(-D) \to 0$$

gives that

$$\det(T(-D)) = [L(-D)^{-1}] = [L(D)].$$

Taking the direct sum of these two maps yields the desired morphism

$$K_0(X) \xrightarrow{\text{ch}} \mathbb{Z} \oplus \operatorname{Pic}(X).$$

Let's now check that this is an isomorphism. This map is easily surjective, the formal difference $(n-1)[O_X] + [L]$ maps onto $(n, [L])$.

To show that this map is injective, it suffices to show that it has no kernel, so we need to show that if $\operatorname{rk}(\mathcal{F}) = \operatorname{rk}(\mathcal{G})$, and $\det(\mathcal{F}) = \det(\mathcal{G})$, then $[\mathcal{F}] = [\mathcal{G}]$ in $K_0(X)$. We will induct on the rank of such an $\mathcal{F}$, noting that it is finite by coherence assumption.

For the base case, if the rank is 0, then $\mathcal{F}$ and $\mathcal{G}$ are torsion, so they are equivalent to $T(D)$ and $T(D')$ for divisors D, D' in the Grothendieck group of torsion coherent sheaves by corollary 4.48. Then, we have

$$[L(-D)] = \det(T(D)) = \det(T(D')) = [L(-D')].$$

So we have that $L(D)$ is isomorphic to $L(D')$, so since $[T(D)] = [O_X] - [L(D)]$, we have that $[T(D)] = [T(D')]$, as desired.

Now for representatives $\mathcal{F}$, $\mathcal{G}$ in $K_0(X)$ of nonzero rank, by tensoring an injective map $O_X \to \mathcal{F} \otimes L$ with L^{-1}, we obtain an injective map from $L \to \mathcal{F}$ and similarly obtain an injective $L' \to \mathcal{G}$. But for any two L, L' we may find an L'' mapping injectively to both of them, so we may assume there exists invertible L'' with an injective map to both $\mathcal{F}$ and $\mathcal{G}$. But then quotienting by the image of L'' results in a coherent sheaf of strictly smaller rank, so $[\mathcal{F}] = [\mathcal{G}]$ by induction, completing the proof.

Remark 4.58. We called the map of Theorem 4.49 ch since it is an algebraic version of the chern character from complex K-theory to rational singular cohomology. This theorem really is the tip of an iceberg beyond the author's knowledge, the interested reader may read more at [Ful84].

4.4 Pushforward on divisors and sheaves

Since the pushforward f_* on coherent sheaves is locally given by restriction, it maps coherent torsion sheaves to coherent torsion sheaves. This induces a natural map of their Grothendieck groups. We now know that these are just the groups of divisors, and we may recognise this induced map as the pushforward on divisors.

Proposition 4.59. *The pushforward on Grothendieck groups is compatible with the pushforward on divisors. That is, for a finite morphism $f : X \to Y$ of Dedekind schemes, the following diagram commutes.*

$$\begin{array}{ccc} K_0(\mathrm{Coh}_{\mathrm{Tor}}(X)) & \xrightarrow{f_*} & K_0(\mathrm{Coh}_{\mathrm{Tor}}(Y)) \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ \mathrm{Div}(X) & \xrightarrow{f_*} & \mathrm{Div}(Y) \end{array}$$

Proof. In view of the direct sum decompositon of both sides, it suffices to show that this map on $T(-Q)$ agrees with the norm map for points Q/P. Since $O_{X,Q}/\mathfrak{q}$ is an $f_{Q/P}$ dimensional vector space over $O_{Y,P}/\mathfrak{p}$, this follows at once. □

Taking the pushforward of invertible sheaves then the determinant yields a "pushforward" operation on Picard groups, and we have the following description of this operation.

Proposition 4.60. *For a finite morphism $f : X \to Y$ of Dedekind schemes, we have an isomorphism*

$$\det(f_*L(D)) \cong \det(f_*O_X) \otimes L(f_*D).$$

Proof. First, for an effective divisor D we have the following short exact sequence.

$$0 \to L(-D) \to O_X \to T(-D) \to 0 \tag{4.1}$$

Note that for any torsion sheaf T, and any line bundle L, we have an isomorphism $T \otimes L \cong T$, since the stalks are unchanged. So tensoring sequence 3.1 with $L(D)$ we obtain

$$0 \to O_X \to L(D) \to T(-D) \to 0 \tag{4.2}$$

Now for an arbitrary divisor D', express $D' = D'' - D$ for D, D'' effective divisors, and tensor sequence 3.1 with $L(D'')$ to obtain

$$0 \to L(D'' - D) \to L(D'') \to T(-D) \to 0. \tag{4.3}$$

So applying f_*, then the determinant to these sequences yields

$$\det(f_*L(D''-D))\otimes L(f_*D)\cong^{1.3}\det(f_*L(D''))\cong^{1.2}\det(f_*O_X)\otimes L(f_*D'')$$

From which rearranging terms yields the result. □

Now we will interpret the different and discriminant of chapter 3 our new tools. Let $f: X\to Y$ be a finite morphism of Dedekind schemes.

Proposition 4.61. *The class of the different divisor $\mathcal{D}_{X/Y}$ in $K_0(\mathrm{Coh}_{\mathrm{Tor}}(X))$ is given by the cokernel of*

$$\mathrm{Tr}^!_{X/Y}: O_X\to f^!O_Y$$

where $\mathrm{Tr}^!_{X/Y}$ is the adjoint of the sheaf trace map $\mathrm{Tr}_{X/Y}$.

Proof. It suffices to check this at points, so at a point Q, the cokernel of this map is generated by a single element, as $(f^!O_Y)_Q$ is free of rank 1 as an $O_{X,Q}$-module. We may realise $\hom_{O_{Y,P}}(O_{X,Q},O_{Y,P})$ as a subset of $K(X)$ via the nondegeneracy of the trace form. So we see that the class of this cokernel at Q is given by the smallest n such that if

$$\mathrm{Tr}(xb)\in O_{Y,P}\text{ for all }b\in O_{X,Q},\text{ then }\pi^n x\in O_{X,P}.$$

This is then seen to be equivalent to the definition of the different at Q given in defintion 3.54. □

We see therefore that the different measures how far the trace form is from being a perfect pairing on the locally free O_Y-module f_*O_X.

We are now in the position to prove Theorem 3.58, which we recall.

Theorem 4.62. *The pushforward of the different divisor $\mathcal{D}_{X/Y}$ is the discriminant divisor $\Delta_{X/Y}$.*

$$f_*\mathcal{D}_{X/Y}=\Delta_{X/Y}.$$

Proof. Applying the pushforward to

$$0\to O_X\to f^!O_Y\to T(-\mathcal{D}_{X/Y})\to 0$$

gives the short exact sequence

$$0\to f_*O_X\to f_*f^!O_Y\to f_*T(-\mathcal{D}_{X/Y})\to 0.$$

Locally on a cover $\{U_i\}_{i\in I}$ we may pick bases of the free modules $f_*O_X(U_i)$ and $f_*f^!O_Y(U_i)$ over $O_Y(U_i)$ such that the map is diagonal with respect to these bases. From this, locally we see that the class of the cokernel is the direct sum of the $O_{Y,P}/d_i$ where d_i are the diagonal elements of our matrix of $\mathrm{Tr}^!_{X/Y}$, which is seen to be equivalent in $K_0(\mathrm{Coh}_{\mathrm{Tor}}(Y))$ to $O_{Y,P}/\det(\mathrm{Tr}^!)$. This determinant was the discriminant as originally defined. So using the isomorphism of $Div(Y)$ and $K_0(\mathrm{Coh}_{\mathrm{Tor}}(Y))$, we have $\Delta_{X/Y} = f_*\mathcal{D}_{X/Y}$. □

We end this section with the observation that the discriminant has a canonical square root.

Proposition 4.63. *As invertible sheaves on Y, we have:*

$$\det(f_* O_X)^2 \cong L(-\Delta_{X/Y})$$

Proof. We have that $f_* f^!(O_Y) \cong \mathrm{Hom}(f_* O_X, O_Y)$, and $\det(\mathrm{Hom}(f_* O_X, O_Y)) = \det(f_* O_X)^{-1}$ by Proposition 4.54. Now taking the determinant of the short exact sequence

$$0 \to f_* O_X \to \mathrm{Hom}(f_* O_X, O_Y) \to f_* T(-\mathcal{D}_{X/Y}) \to 0$$

we obtain

$$\det(f_* O_X)^2 \cong L(-f_*\mathcal{D}_{X/Y}) \cong L(-\Delta_{X/Y}).$$

□

Chapter 5

Curves

5.1 Curves and function fields

In this section we will be looking at a classical class of Dedekind schemes, curves over a field k.

Definition 5.1 (Curve over a field)**.** A curve X over a field k is a Dedekind scheme X with a morphism $X \xrightarrow{s_X} \text{Spec}(k)$ such that $O_X(X) = k$, with $O_X(U_i)$ a finitely generated k algebra for an open affine cover $\{U_i\}_{i \in I}$ of X. A morphism of curves $f : X \to Y$ is a morphism of them as Dedekind schemes such that $s_Y \circ f = s_X$.

Remark 5.2. One may show that $O_X(U_i)$ being a finitely generated k algebra on an open affine cover U_i is equivalent to $O_X(U)$ being a finitely generated k algebra for all open affine U, see exercises 2.3.1 – 2.3.4 in [Har77].

This field k will be present throughout our discussion, so we will drop reference to the maps s_X.

Remark 5.3. Usually, what we define a curve over k to be is called a connected nonsingular complete curve over k, as in [Lor96]. All our Dedekind schemes are connected and nonsingular, and since we will only consider complete curves, we have chosen to suppress these adjectives.

Example 5.4. The Dedekind scheme $\mathbb{P}^1_k$ is a curve over k. We saw in chapter 2 that its global sections were precisely the field k, and $K(\mathbb{P}^1_k) \cong k(t)$. Formally, the structure map is given by the constant morphism to Spec(k), with the map $k = O_{\mathbb{P}^1_k}(\mathbb{P}^1_k)$.

Throughout this these, we have seen a close relationship between Dedekind schemes and their fields of fractions. In the case of curves, the relationship is as close as possible, we may reconstruct a curve from its field of fractions. We say that a subfield $k \subset L$ is algebraically closed in L if any element of L algebraic over k is contained in k.

Definition 5.5 (Function field over k)**.** A function field L over k is a finitely generated field extension L/k of transcendence degree 1, with k algebraically closed in L. A morphism of function fields is a finite field extension $L \xrightarrow{\iota} L'$ that the following diagram commutes.

$$\begin{array}{ccc} L & \xrightarrow{\iota} & L' \\ \uparrow & \nearrow & \\ k & & \end{array}$$

We wish to show that function fields over k correspond to curves over k. First, we need the following technical lemma, which we prove following the argument of theorem X.1.7 of [Lor96].

Lemma 5.6. *If $L/k(x)$ is a finite extension of fields, then the integral closure of $k[x]$ in L is a finitely generated $k[x]$ module.*

Proof. Let's first prove this in the case of a purely inseparable extension. For $p = \mathrm{char}(k)$, a finite purely inseparable extension L of $k(x)$ is of the form

$$L = k(x)(q_1^{1/p^{n_1}}, q_2^{1/p^{n_2}}, ., q_m^{1/p^{n_m}})$$

where $q_i \in k(x)$. This is contained within the finite extension $L' = k'(x)(x^{1/p^N})$ where $N = \max_{1 \le i \le m}\{n_i\}$, and k' is the finite extension of k obtained by adjoining the p^Nth roots of all the coefficients of $q_1, ., q_m$. The integral closure of $k[x]$ in L is contained in the integral closure of $k[x]$ in L', and this is just $k'[x^{1/p^N}]$, which is finitely generated as a $k[x]$ module, so the integral closure is finitely generated.

Now for an arbitrary extension L, we may without loss of generality take L to be normal over $k(x)$. This is because if we show the integral closure is finitely generated in a finite extension of L, then since $k[x]$ is Noetherian, we will have shown our desired integral closure is finitely generated. In this situation, taking M to be the largest inseparable subextension of L containing $k(x)$, we have that L/M is separable, and $M/k(x)$ purely inseparable, see proposition 6.1 in chapter V of [Lan02]. By the previous result, the integral closure B of $k[x]$ in M is finitely generated over $k[x]$, and by Theorem 3.16, the integral closure C of B in L is finitely generated over B, so C is finitely generated over $k[x]$, as desired. □

Proposition 5.7. *For X a curve over k, the fraction field $K(X)$ is a function field over k.*

Proof. Pick some affine U in X. Since $O_X(U)$ is a domain that is a finitely generated k algebra, its Krull dimension equals the transcendence degree of its fraction field over k, see theorem 11.25 of [AM69]. To see that k is algebraically closed in $K(X)$, note that any x algebraic over k must have $v_P(x) = 0$ for all points P. To see this, apply v_P to an equation witnessing its algebraicity, and use the elementary properties of valuations given in definition 2.49. Thus, any x algebraic over k will be contained in

$$\bigcap_{P \in X} O_{X,P} = O_X(X) = k.$$

□

Conversely, for any function field over k, we may construct a locally ringed space, which will end up being a curve over k, in a manner which by now should look familiar.

Construction 5.8. For a function field L over k, we construct a locally ringed space $(V_k(L), O_{V_k(L)})$ as follows:

- The points of $V_k(L)$ are equivalence classes of valuations on L that vanish on k.
- The topology on $V_k(L)$ is generated by setting each nonzero equivalence class of valuations to be closed.
- The structure sheaf is given by

$$O_{V_k(L)}(U) := \bigcap_{v \in U} O_v = \{x \in L | v(x) \geq 0 \text{ for all } v \in U\}.$$

Let's now check that for $k(x)/k$, the associated locally ringed space is $\mathbb{P}^1_k$. For this, we just need to check that every valuation on $k(x)$ which vanishes on k is associated to a prime ideal in $k[x]$ or v_∞.

Proposition 5.9. *Every nonzero valuation on $k(x)$ that vanishes on k is equivalent to v_p or v_∞, for p an irreducible polynomial of positive degree in $k[x]$.*

Proof. For a nonzero valuation v, assume first that it is negative on some polynomial $p(x)$, which we may take to have minimal degree with this property. Expressing $p(x)$ as

$$p(x) = xq(x) + a \text{ for } a \in k$$

we see that the degree of $p(x)$ is 1 by minimality, and v is equivalent to v_∞. So now take v to be positive on some polynomial $p(x)$. From the previous analysis, we see v is nonnegative on all polynomials. We see that v is therefore positive on some irreducible factor $q(x)$ of $p(x)$. Now if v was not equivalent to v_q, then it would be positive on some other irreducible polynomial $r(x)$ of positive degree. But since $r(x)$ and $q(x)$ generate the unit ideal, applying v to the expression

$$r(x)a(x) + q(x)b(x) = 1$$

yields a contradiction. So $v \sim v_q$, and we have proved the claim in all cases. □

We are now ready to prove our desired equivalence.

Theorem 5.10. *The functor taking a curve X over k to $K(X)$ yields an equivalence of categories between curves over k with finite maps, and function fields over k. The pseudoinverse of this functor is given by taking the locally ringed space $V_k(L)$ associated to a function field L over k.*

Proof. Let's first check that $V_k(L)$ is a curve over k. Pick an element $t \in L \backslash k$, with corresponding finite extension $L/k(t)$. Frst note that any nonzero valuation v on L restricts to a nonzero valuation on $k(t)$, since $L/k(t)$ is an algebraic extension. To see this, if $v(x) > 0$, then apply v to an equation witnessing the algebraicity of x over $k(t)$, and use the elementary properties of definition 2.49. So since every valuation on $k(t)$ is positive on an open affine U of $\mathbb{P}^1_k$ from Proposition 5.9, we see that $V_k(L) = v_X(L)$, our Construction 2.60. So we have a Dedekind scheme $V_k(L)$, and a map $V_k(L) \to \mathbb{P}^1_k$, with each $O_{V_k(L)}(f^{-1}(U))$ the integral closure of $\mathbb{P}^1_k(U)$ in L. Take

$$U = \operatorname{Spec}(k[t]) = \mathbb{P}^1_k \setminus v_0$$

and

$$U' = \operatorname{Spec}(k[t^{-1}]) = \mathbb{P}^1_k \setminus v_\infty.$$

to be our open cover of $\mathbb{P}^1_k$. Corollary 3.6 then gives that the following as an open affine cover of $V_k(L)$.

$$V := f^{-1}(U), V' := f^{-1}(U')$$

On these open sets, $O_{V_k(L)}(V)$ is finitely generated as a $k[t]$ module, and similarly with V'. So the morphism $V_k(L) \to \mathbb{P}^1_k$ is finite, and since $O_{V_k(L)}(V_k(L))$ is the integral closure of $O_{\mathbb{P}^1_k}(\mathbb{P}^1_k) = k$ in L, we have that $V_k(L)$ is a curve.

So we see that the functor is fully faithful, and essentially surjective, with pseudoinverse $L \to V_k(L)$. □

5.2 Degree of a point on a curve

The presence of a base field k allows for an absolute measure on the size of our points, given by the degree of the residue extension $\kappa(P)/k$.

Definition 5.11 (Degree of a point)**.** The degree $\deg_X(P)$ of a closed point P in X is the degree of the field extension $\kappa(P)/k$.

For this to make sense, we need to check that this degree is finite. Given $P \in X$, let π be a uniformiser at P, and B be the integral closure of $k[\pi]$ in $K(X)$. Since $k(\pi) \to K(X)$ is a finite extension, the associated map of curves $X \to \mathbb{P}^1_k$ is finite. Therefore $f_{P/(\pi)}$ is finite, and this is exactly $\deg_X(P)$.

We define the degree of a divisor similarly.

Definition 5.12 (Degree of a divisor)**.** The degree of a divisor $D = \sum_{P\in X} a_P P$ in $\operatorname{Div}(X)$ is given by

$$\deg_X(D) := \sum_{P\in X} a_P \deg_X(P) = \sum_{P\in X} a_P[\kappa(P) : k].$$

Proposition 5.13. *The degree respects our operations on divisors. Let $f : X \to Y$ be a finite map of curves over k, with $D \in \operatorname{Div}(X)$, and $D' \in \operatorname{Div}(Y)$. Then we have*

$$\deg_Y(f_*D) = \deg_X(D)$$

and

$$\deg_X(f^*D') = [K(X) : K(Y)] \deg_Y(D').$$

Proof. By linearity, it suffices to prove this for divisors of the form Q. The first claim follows since the extension $\kappa(Q)/k$ is the composition of the two extensions $\kappa(Q)/\kappa(f(Q))$ and $\kappa(f(Q))/k$. For the second, note that $\deg_X(f^*D') = \deg_Y(f_*f^*D')$ from the first proposition, and recall from Proposition 3.48 that we have

$$f_*f^*D' = [K(X) : K(Y)]D'.$$

□

Proposition 5.14 (Degree of a principal divisor is zero). *Let X be a curve over k. For any $f \in K(X)^*$, we have* $\deg_X(\mathrm{div}(f)) = 0$.

Proof. First, let's prove this for $\mathbb{P}^1_k$. Since $k[x]$ is a unique factorisation domain, and

$$\deg_X(\mathrm{div}(fg)) = \deg_X(\mathrm{div}(f)) + \deg_X(\mathrm{div}(g))$$

it suffices to show the claim for an irreducible polynomial $p(x) \in k[x]$. We see that if $\mathfrak{p}$ is the ideal generated by $p(x)$, then $v_{\mathfrak{p}}(p(x)) = 1$, and $v_\infty(p(x)) = -n$, where n is the degree of $p(X)$ as a polynomial. But the degree of v_∞ is 1, and the degree of the point P is n, so this sum is 0. For the case of a general curve X, pick a transcendental element $x \in K(X)$, with associated finite map $X \to \mathbb{P}^1_k$. By propositions 5.14 and 5.13 we therefore have

$$\deg_X(\mathrm{div}(f)) = \deg_{\mathbb{P}^1_k}(f_* \mathrm{div}(f)) = \deg_{\mathbb{P}^1_k}(\mathrm{div}(\mathrm{N}_{X/\mathbb{P}^1_k}(f))) = 0.$$

□

From this, we see that the class group has a canonical map to $\mathbb{Z}$.

Corollary 5.15 (Degree as a map on class groups). *The degree map on divisors extends to a map* $\mathrm{Cl}(X) \to \mathbb{Z}$.

5.3 Adele ring of a curve

We saw in chapter 3 that completing the local rings $O_{X,P}$ led to better linear alegrbaic properties. For a given curve X, we will construct a ring which completes at all points of X at once, allowing us to use the desirable properties of the completion in a global setting. First, we define the ring $\widehat{K(X)}^P$ to be $K(X) \otimes_{O_{X,P}} \widehat{O}^{\mathfrak{p}}_{X,P}$. This is also seen to be the quotient ring of $\widehat{O}^{\mathfrak{p}}_{X,P}$, since we only need to invert a uniformiser to obtain a field, which can be taken to lie in $K(X)$ by Lemma 3.25.

Definition 5.16 (Adele ring of a curve). For a curve X/k, the ring of adeles $\mathbb{A}_X$ of X is defined to be the subring of the product ring $\prod_{P \in X} \widehat{K(X)}^P$, where all but finitely many components are contained in $\widehat{O}^{\mathfrak{p}}_{X,P}$.

We will write an adele as a sequence $(\alpha_P)_{P \in X}$ indexed by points of X, with $\alpha_P \in \widehat{K(X)}^P$, such that $\alpha_P \in \widehat{O}^{\mathfrak{p}}_{X,P}$ for all but finitely many points $P \in X$.

We have a natural diagonal map $K(X) \xrightarrow{\Delta} \mathbb{A}_X$ given by taking the product of the maps

$$K(X) \to K(X) \otimes_{O_{X,P}} \widehat{O}^{\mathfrak{p}}_{X,P} = \widehat{K(X)}^P.$$

This can give us some intuition for how to think about the adeles. At each point P of X, a function f on X gives a sequence of infinitesmal data, see the discussion following definition 3.19. By completing at a point P, we consider all sequences of infinitesmal data at P, and the adele ring is the global version of this, considering the collections of all infinitesmal local data at once, regardless of whether they arise from a global function.

Our valuations on $K(X)$ naturally extend to valuations on $\mathbb{A}_X$, we define v_Q by

$$v_Q((\alpha_P)_{P \in X}) = v_Q(\alpha_Q).$$

Remark 5.17. This is the definition of the adele ring of a curve, and a similar construction occurs in number theory for $\mathrm{Spec}(O_K)$. This is slightly different however, one also needs to take into account all the completions of K, such as the real numbers $\mathbb{R}$ for $\mathrm{Spec}(\mathbb{Z})$.

For any open subset U of X, we have a distinguished subring of $\mathbb{A}_X$, extending $O_X(U)$ along the diagonal embedding.

Definition 5.18. For an open subset U of X, we define

$$\tilde{O}_X(U) = \{(a_P)_{P \in X} \in \mathbb{A}_X | v_P(a_P) \geq 0 \text{ for all } P \in U\}.$$

That is, $\tilde{O}_X(U)$ is those adeles which look like $O_X(U)$ from the perspective of v_P for $P \in U$.

Taken together, these yield a sheaf of rings on X, which we denote by $\tilde{O}_X$. This sheaf of rings can be thought of a completed version of the structure sheaf of X. Its stalks at closed points are now the completed local rings $\widehat{O}^{\mathfrak{p}}_{X,P}$, and its stalk at the generic point is $\mathbb{A}_X$, with $\tilde{O}_X(U) \cap K(X) = O_X(U)$. So we obtain a new locally ringed space, $(X, \tilde{O}_X)$.

Proposition 5.19 (Functoriality of the adele ring)**.** *For a finite map of curves $f : X \to Y$ over k, the natural inclusions $K(X) \to \mathbb{A}_X$ and $\mathbb{A}_Y \to \mathbb{A}_X$ yield a canonical isomorphism*

$$\mathbb{A}_X \cong \mathbb{A}_Y \otimes_{K(Y)} K(X).$$

Proof. First, observe that by corollary 3.27, for a point $P \in Y$, we have the following isomorphisms.

$$\begin{aligned} K(X) \otimes_{K(Y)} \widehat{K(Y)}^P &\cong \\ K(X) \otimes_{K(Y)} K(Y) \otimes_{O_{Y,P}} \widehat{O}^{\mathfrak{p}}_{Y,P} &\cong \\ K(X) \otimes_{O_{Y,P}} \widehat{O}^{\mathfrak{p}}_{Y,P} &\cong \\ K(X) \otimes_{(f_* O_X)_P} (f_* O_X)_P \otimes_{O_{Y,P}} \widehat{O}^{\mathfrak{p}}_{Y,P} &\cong \\ K(X) \otimes_{(f_* O_X)_P} \prod_{Q:Q \to P} \widehat{O}^{\mathfrak{q}}_{X,Q} &\cong \prod_{Q:Q\to P} \widehat{K(X)}^Q \end{aligned}$$

We see therefore that by picking a basis $\{b_i\}_{i=1}^n$ of $K(X)$ over $K(Y)$, we obtain a unique expression of any element

$$(\alpha_Q)_{Q \in X} \text{ in } \prod_{Q \in X} \widehat{K(X)}^Q$$

as

$$(\alpha_Q) = \sum_{i=1}^n b_i (\beta_P)_i$$

where $(\beta_P)_i$ are in $\prod_{P \in Y} \widehat{K(Y)}^P$. Since the b_i have $v_Q(b_i) = 0$ for all but finitely many $Q \in X$, we see that $(\alpha_Q)_{Q \in X}$ will be in $\mathbb{A}_X$ if and only if each $(\beta_P)_i$ is in $\mathbb{A}_Y$. □

Definition 5.20. By tensoring the $K(Y)$ linear trace $\mathrm{Tr}_{K(X)/K(Y)} : K(X) \to K(Y)$ with $\mathbb{A}_Y$, we obtain the trace on adeles

$$\mathrm{Tr}_{X/Y} : \mathbb{A}_X \to \mathbb{A}_Y.$$

Note that this is also given by sum of the local traces, with respect to semilocal decompositions of Proposition 5.19.

We saw in definition 4.36 that for any divisor D on X, we obtained an invertible subsheaf $L(D)$ of $K(X)$. We may also upgrade this to our rings of adeles.

Definition 5.21. For D a divisor on X, we have a sheaf of $\tilde{O}_X$ modules $\tilde{L}(D)$ on X given by

$$\tilde{L}(D)(U) = \{(a_P)_{P \in X} \in \mathbb{A}_X | v_P(a_P) + v_P(D) \geq 0 \text{ for all } P \in U\}.$$

This is the same description of $L(D)(U)$, but intepreted in the adele ring instead of $K(X)$.

Like our other adelic constructions, we have $\tilde{L}(D)(U) \cap K(X) = L(D)(U)$. We wont be utilising these locally free sheaves to the extent of chapter 4, we will be mainly be dealing with their global sections. We will denote the global sections of $\tilde{L}(D)$ by L_D.

5.4 Riemann-Roch, numerical form

In this section, we will prove the Riemann-Roch theorem, a fundamental result describing the behaviour of an invertible sheaf $L(D)$ on X. We will see how the adelic language enables a simple proof of this theorem.

For any divisor D on X, consider the following exact sequence, where Δ_D is the diagonal, followed by the projection onto the quotient.

$$0 \to \ker(\Delta_D) \to K(X) \xrightarrow{\Delta_D} \mathbb{A}_X/L_D \to \operatorname{coker}(\Delta_D) \to 0$$

Definition 5.22. For any divisor D on X, we define two associated k vector spaces;

$$H^0(D) := \ker(\Delta_D)$$

and

$$H^1(D) := \operatorname{coker}(\Delta_D).$$

Observe that since $L_D \cap K(X) = L(D)(X)$, the space $H^0(D)$ is the k vector space of global sections of $L(D)$.

With this, we are ready to state the first form of the theorem. Throughout, $dim_k V$ will denote the k dimension of a finite k vector space V.

Theorem 5.23 (Riemann-Roch, numerical form)**.** *Let X be a curve over k. For any divisor D on X, we have that $H^0(D)$ and $H^1(D)$ are finite dimensional over k, and the integer*

$$\dim_k H^0(D) - \dim_k H^1(D) - \deg_X(D)$$

is independent of D, depending only on X.

To prove this, we will first observe some simple facts.

Lemma 5.24. *If $\deg_X(D) < 0$, then $H^0(D)$ is the zero vector space.*

Proof. For $H^0(D)$ to be nonzero, we require $\operatorname{div}(f) + D \geq O$ for some nonzero $f \in K(X)^*$. By applying $\deg_X$, and using Proposition 5.14, we see $\deg_X(D)$ must be nonnegative. □

Lemma 5.25. *For any divisor $D \in \mathrm{Div}(X)$, the kernel of the natural map*

$$\mathbb{A}_X/L_D \to \mathbb{A}_X/L_{D+P}$$

is isomorphic to $\kappa(P)$, and hence has k dimension $\deg_X(P)$.

Proof. The only coordinate of which this kernel is nonzero is P, and there we see the kernel is $\mathfrak{p}^{v_P(D)}/\mathfrak{p}^{v_P(D)+1} \cong \kappa(P)$. □

Lemma 5.26. *Let $f : X \to Y$ a finite map of curves of degree n, with $g \in K(X)^*$. There exists a divisor D on Y such that the multiplication by g map on adeles descends to a map of quotients μ_g, where O denotes the zero divisor on X.*

$$\begin{array}{ccc} \mathbb{A}_Y & \xrightarrow{(\alpha_P)\to(g\cdot\alpha_P)} & \mathbb{A}_X \\ \downarrow & & \downarrow \\ \mathbb{A}_Y/L_D & \xrightarrow{\mu_g} & \mathbb{A}_X/L_O \end{array}$$

Proof. We give this divisor D as

$$v_P(D) = \min_{Q:Q\to P}\{v_Q(g), 0\}.$$

Let Q/P be points of X and Y, with Q over P. If $0 \le v_P(\alpha_P) + v_P(D)$, then $0 \le v_P(\alpha_P)$, so we have:

$$0 \le e_{Q/P}v_P(\alpha_P) + v_P(D) \le e_{Q/P}v_P(\alpha_P) + v_Q(g) \le v_Q(g\cdot\alpha_P)$$

From which the claim follows. □

We will need one final lemma, which is the core fact at the heart of the proof.

Lemma 5.27. *For the zero divisor O on $\mathbb{P}^1_k$, we have $H^1(O) = 0$. That is, every adele (α_P) can be expressed as*

$$(\alpha_P) = f + (\beta_P) \tag{5.1}$$

where $f \in K(X)$, and $v_P(\beta_P) \ge 0$ for all $P \in X$.

Proof. For any adele (a_P), for all but finitely many components, α_P is contained in $\widehat{O}^{\mathfrak{p}}_{\mathbb{P}^1_k,P}$, so it suffices to show that every adele supported at a single point P can be expressed in this form of (5.1). By applying the $t \to t^{-1}$ automorphism of $\mathbb{P}^1_k$, we may assume that the point P is not the point at infinity, and corresponds to a prime ideal of $k[t]$. Since $k[t]$ is a principal ideal domain, any prime ideal $\mathfrak{p}$ is generated by a unique monic irreducible polynomial π.

So if G_P is the adele supported only at P by the element $G \in \widehat{k(\mathbb{P}^1)}^P$, then for some $m \in \mathbb{N}$ we have $v_P(\pi^m G_P) \geq 0$. The coset of $\pi^m G_P$ in $k[t]/\pi^m$ has a unique polynomial representative q of degree less than $m \cdot \deg(\pi)$. We claim that

$$G_P = q/\pi^m + (G_P - q/\pi^m)$$

gives a decomposition of the form (5.1). First, we see that for $Q \neq P$ a point corresponding to a prime ideal of $k[t]$, we have

$$v_Q(G_P - q/\pi^m) \geq v_Q(q/\pi^m) \geq 0.$$

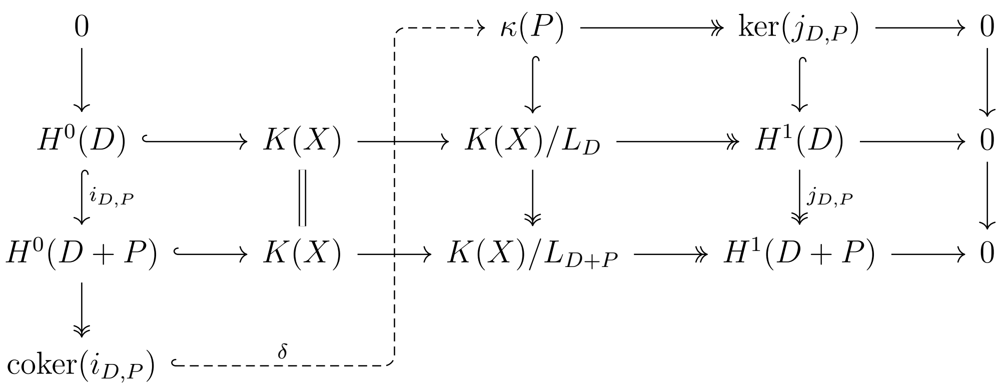
At infinity, since $\deg(q) < m \cdot \deg(\pi)$, we also have

$$v_\infty(q/\pi^m) \geq 0.$$

Finally, at P, by construction, we have $v_P(G - q/\pi^m) \geq 0$. $\square$

We can now prove the theorem.

Proof of Theorem 5.23. Consider the following diagram:



From this, we obtain three short exact sequences.

$$0 \to H^0(D) \to H^0(D+P) \to \mathrm{coker}(i_{D,P}) \to 0 \tag{5.2}$$

$$0 \to \ker(j_{D,P}) \to H^1(D) \to H^1(D+P) \to 0 \tag{5.3}$$

$$0 \to \mathrm{coker}(i_{D,P}) \xrightarrow{\delta} \kappa(P) \to \ker(j_{D,P}) \to 0 \tag{5.4}$$

Only the last of these requires an explanation. To define the map δ, lift an element of the cokernel and include it into $K(X)$, then project it into $K(X)/L_D$. By commutativity, this will land in the image of $\kappa(P)$, and one may verify by a simple diagram chase that this map is well defined and the sequence is exact.

So from this, we see that since $\kappa(P)$ is finite dimensional, $\operatorname{coker}(i_{D,P})$ and $\ker(j_{D,P})$ are also finite dimensional. Observe that any divisor D can be obtained from any other divisor D' by a finite sequence of adding and removing points of X. The finite dimensionality of these differences therefore implies that if for some divisor $D \in \operatorname{Div}(X)$, $\dim_k H^i(D)$ is finite, then $\dim_k H^i(D')$ is also finite for all $D' \in \operatorname{Div}(X)$.

Since $\dim_k H^0(O) = 1$ for all curves directly from our definition 5.1, we see that $H^0(D)$ is finite dimensional for all divisors D on a curve X.

In the specific case of $\mathbb{P}^1_k$, since $\dim_k H^1(O) = 0$ by Lemma 5.27, we get that $H^1(D)$ is finite dimensional for all divisors D on $\mathbb{P}^1$. Now for an arbitrary curve, pick a finite morphism to $\mathbb{P}^1_k$, and $\{g_i\}_{i=1}^n$ a basis of $K(X)/K(\mathbb{P}^1)$.

Then from lemma5.26 there exist $D_i \in \operatorname{Div}(\mathbb{P}^1_k)$ such that we have the following diagram

$$\begin{array}{ccccccc}
\bigoplus_{i=1}^n K(\mathbb{P}^1) & \longrightarrow & \bigoplus_{i=1}^n \mathbb{A}_{\mathbb{P}^1}/L_{D_i} & \longrightarrow & \bigoplus_{i=1}^n H^1(D_i) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \sum \mu_{g_i}} & & \downarrow{\scriptstyle \sum \mu_{g_i}} & & \downarrow & & \\
K(X) & \longrightarrow & \mathbb{A}_X/L_O & \longrightarrow & H^1(O) & \longrightarrow & 0
\end{array}$$

Where the maps $\sum \mu_{g_i}$ are surjective by Proposition 5.19. Thus, the induced map

$$\bigoplus_{i=1}^n H^1(D_i) \to H^1(O)$$

is surjective. So $H^1(O)$ is finite dimensional for an arbitrary curve X.

So now on an arbitrary curve X, the finite dimensionality of $\ker(j_{D,P})$ gives that $\dim_k H^1(D)$ is finite for all divisors $D \in \operatorname{Div}(X)$.

So all the vector spaces in sequences 5.2,5.3 and 5.4 are finite dimensional, so since dimension is additive in short exact seqences, we have

$$\dim_k H^0(D) - \dim_k H^1(D) = \dim_k H^0(D+P) - \dim_k H^1(D+P) - \dim_k(\kappa(P))$$

So since $\deg_X(P) = \dim_k(\kappa(P))$, we see that this quantity

$$\dim_k H^0(D) - \dim_k H^1(D) - \deg_X(D)$$

is independent of the divisor D, completing the proof. □

Definition 5.28. For a curve X over k, we call the integer $\dim_k H^1(O)$ the genus of X, denoted g_X.

Corollary 5.29. *We may phrase Theorem 5.23 as the equality*

$$\dim_k H^0(D) - \dim_k H^1(D) = \deg_X(D) - g_X + 1$$

Proof. This holds for $D = O$, and from Theorem 5.23, this number $1 - g_X$ is independent of the divisor D. □

This terminology is not a coincidence, in the case of curves over $\mathbb{C}$, the closed points of X can be given the structure of a real two dimensional manifold, and the genus of X as a curve agrees with the topological genus of this surface, though this is a deep result, see [Mir95].

5.5 Weil differentials

The full Riemann-Roch theorem is more than just the numerical statement of Theorem 5.23, we can actually interpret the space $H^1(D)$ as the sections of another invertible sheaf, with Riemann-Roch becoming a statement of duality. We will use Weil differentials to prove this, since they allow for a simple rapid proof.

While technically easier, this approach obscures the geometric aspect of this theorem. The reader familiar with differential forms is invited to read Theorem 5.47 to see the relation of Weil differentials with the usual sheaf of differentials.

Definition 5.30 (Weil differentials)**.** A Weil differential on X is a k linear functional on $\mathbb{A}_X/K(X)$ that vanishes on L_D for some divisor D.

We denote the set of Weil differentials that vanish on L_D by Ω_D. Note that the k dimension of Ω_D is $\dim_k H^1(D)$, since this is the k dual of $H^1(D)$. For divisors $D \leq D'$, since $L_D \subset L_{D'}$ we have $\Omega_{D'} \subset \Omega_D$.

Proposition 5.31. *The set of Weil differentials can be given the structure of a $K(X)$ vector space, denoted $J(X)$. With this structure, we have*

$$f \cdot \Omega_D \subset \Omega_{D-\mathrm{div}(f)}.$$

Proof. If ω is a Weil differential, then the functional $f \cdot \omega((a_i)) = \omega(f \cdot (a_i))$ is a Weil differential, since if ω vanishes on L_D, then $f\omega$ vanishes on $L_{D-\mathrm{div}(f)}$.

It remains to show that Weil differentials are closed under addition. If ω_1 and ω_2 vanish on L_{D_1} and L_{D_2} respectively, then $w_1 + w_2$ vanishes on D_3 where D_3 is defined by $v_P(D_3) = \min\{v_P(D_1), v_P(D_2)\}$. □

Definition 5.32. For any $\omega \in J(X)$, we define its divisor $\operatorname{div}(\omega)$ to be the maximal D such that $\omega \in \Omega_D$.

Note that this is well defined, since if $\omega \in \Omega_{D_1} \cap \Omega_{D_2}$, then $\omega \in \Omega_{D_3}$ where $v_P(D_3) = \max\{v_P(D_1), v_P(D_2)\}$.

From Proposition 5.31, we have that $\operatorname{div}(f \cdot \omega) = \operatorname{div}(f) + \operatorname{div}(\omega)$.

Theorem 5.33. *The space of Weil differentials $J(X)$ is a one dimensional vector space over $K(X)$.*

Proof. Let ω_1 and ω_2 be Weil differentials. Pick a sufficiently negative divisor D such that $\omega_1, \omega_2 \in \Omega_D$. Then for any divisor D', we have a map

$$H^0(D') \bigoplus H^0(D') \to \Omega_{D-D'} \tag{5.5}$$

given by mapping (f, g) to $f\omega_1 + g\omega_2$. This is k linear, and well defined by Proposition 5.31.

Now note from corollary 5.29 and Lemma 5.24, that if D' has $\deg_X(D - D') < 0$, then

$$\dim_k \Omega_{D-D'} = \dim_k H^1(D - D') = \deg_X(D') - \deg_X(D) + g_X - 1.$$

From corollary 5.29, we also have the following inequality, classically known as Riemann's inequality.

$$\dim_k H^0(D') \geq \deg_X(D') - g_X + 1$$

So now we see for sufficiently large divisors D', those with

$$\deg_X(D') > 3g_X - 3 - \deg_X(D)$$

we have $2 \dim_k H^0(D') > \Omega_{D-D'}$. So in this situation, our map 5.5 is not injective. Thus there exist $f, g \in K(X)^*$ with $f\omega_1 = g\omega_2$, so the space $J(X)$ is one dimensional. $\square$

In view of Theorem 5.33, we see that the divisor class of $\operatorname{div}(\omega)$ in $\operatorname{Cl}(X)$ is independent of the divisor chosen. We call this divisor class the canonical class K_X. By abuse of notation, we will also use K_X to denote a (noncanonically chosen) divisor in this class.

5.6 Riemann-Roch, duality form

We now have all the groundwork laid for the full Riemann-Roch theorem.

Theorem 5.34 (Riemann-Roch)**.** *The canonical evaluation pairing*

$$J(X) \otimes_{K(X)} \mathbb{A}_X \to k$$

induces a nondegenerate pairing $\Omega_D \otimes_k H^1(D) \to k$. Picking a Weil differential ω with divisor K_X, we obtain an isomorphism $\Omega_D \cong H^0(K_X - D)$, and so we have

$$H^0(K_X - D) \cong H^1(D)^*.$$

Thus obtaining the duality form of the Riemann-Roch theorem.

$$\dim_k H^0(D) - \dim_k H^0(K_X - D) = \deg_X(D) - g_X + 1.$$

Proof. The nondegeneracy of the pairing $\Omega_D \otimes_k H^1(D) \to k$ follows from our definition of Ω_D. If ω is a chosen Weil differential, then $H^0(K_X - D) \cdot \omega \subset \Omega_D$. From Theorem 5.33, and Proposition 5.31 this inclusion is an equality, giving the result. □

We will now investigate some of the corollaries of Theorem 5.34.

The first is that once we go above a certain degree, the dimension of the global sections of $L(D)$ grows uniformly with the degree.

Proposition 5.35. *If D is a divisor on X, with $\deg_X(D) > 2g_X - 2$, then*

$$\dim_k H^0(D) = \deg_X(D) - g_X + 1.$$

Proof. For D of this degree, we have $\deg_X(K_X - D) < 0$, so the space $H^0(K_X - D)$ is zero dimensional by 5.24. □

The upshot of this proposition is that the genus of a curve X over k can be calculated from the numerical quantities of $H^0(D)$ and $\deg_X(D)$ as D gets large. This, with some Galois theory allows one to show that the genus of the curve associated to the function field $K(X') := K(X) \otimes_k k'$ equals the genus of X. For a proof of this, see corollary $IX.5.8$ of [Lor96].

The next proposition shows that a canonical divisor K_X is quite easy to find, as it is determined by simple numerical data.

Proposition 5.36. *For X a curve over k, the degree of K_X is $2g_X - 2$, and $H^0(K_X)$ is g_X dimensional. Any divisor D with these properties is also a canonical divisor, for some $f \in K(X)^*$, we have $K_X = D + \operatorname{div}(f)$.*

Proof. We have from Theorem 5.34

$$\dim_k H^0(K_X) = \dim_k H^1(O) := g_X$$

and

$$\dim_k H^0(O) = \dim_k H^1(K_X) = 1.$$

By Theorem 5.23, we have

$$\dim_k H^0(K_X) - \dim_k H^1(K_X) - \deg_X(K_X) = 1 - g_X$$

so the degree of K is $2g_X - 2$. Conversely, let D be a divisor of degree $2g_X - 2$ with $\dim_k H^0(D) = g_X$. We have

$$\dim_k H^0(D) - \dim_k H^0(K_X - D) - (2g_X - 2) = 1 - g_X.$$

So we see that $H^0(K_X - D)$ is one dimensional, and $K_X - D$ is degree 0. For $f \in H^0(K_X - D)$, we have $\operatorname{div}(f) + K_X - D \geq O$, but this divisor is degree 0, hence is the zero divisor O, giving the result. □

Our last corollary requires some more theory first, we will need to interpret Weil differentials locally.

Definition 5.37. The component of a Weil differential ω at P is the k linear functional ω_P given by the composite

$$\widehat{K(X)}^P \to \mathbb{A}_X \to \mathbb{A}_X/K(X) \xrightarrow{\omega} k.$$

Using these local components, we may describe the P component of the divisor $\operatorname{div}(\omega)$ as follows.

Proposition 5.38. *Let P be a point of X, with $\mathfrak{p}$ its associated maximal ideal of $\widehat{O}^{\mathfrak{p}}_{X,P}$. The functional ω_P vanishes on $\mathfrak{p}^{-v_P(\operatorname{div}(\omega))}$, and does not vanish on $\mathfrak{p}^{-v_P(\operatorname{div}(\omega))-1}$.*

Proof. We know that ω_P vanishes on any $x \in \mathfrak{p}^{-v_P(\operatorname{div}(\omega))}$, since it vanishes on the adele $\bar{x}$ supported only at P. For the nonvanishing, we know that ω doesnt vanish on $L_{\mathrm{div}(\omega)+P}$, so there exists an adele (α_Q) with $\omega((\alpha_Q)) \neq 0$. We may express this as

$$(\alpha_Q)_{Q\in X} = \bar{\alpha_P} + (\alpha_Q)_{Q\neq P\in X}.$$

Here $\bar{\alpha}_P$ is the adele with component α_P only at P. Since $(\alpha_Q)_{Q\neq P\in X} \in L_{\mathrm{div}(\omega)}$, by linearity, we have $\omega_P(\alpha_P) \neq 0$, and we know that $\alpha_P \in \mathfrak{p}^{-v_P(\mathrm{div}(\omega))-1}$ by construction. □

From this, we see that an arbitrary k linear functional ϕ on $\mathbb{A}_X$ will be a Weil differential if and only if it satisfies the following properties:

i) For all but finitely many points $P \in X$, each component ϕ_P vanishes on $\widehat{O}^{\mathfrak{p}}_{X,P} \subset \widehat{K(X)}^P$, and does not vanish on some $x \in \widehat{O}^{\mathfrak{p}}_{X,P}$ with $v_P(x) = -1$.
ii) For all the other points $Q \in X$, ϕ_Q vanishes on $\widehat{\mathfrak{q}}^n \subset \widehat{O}^{\mathfrak{q}}_{X,P}$ for some $n \in \mathbb{Z}$.
iii) The functional ϕ vanishes on $K(X) \subset \mathbb{A}_X$.
iv) The component ϕ_P is not identically 0 for any $P \in X$.

We will now define the trace on Weil differentials.

Definition 5.39. Let $f : X \to Y$ be a finite, separable morphism of curves over k. We define the trace map on Weil differentials by

$$\mathrm{Tr}\,\omega := \omega \circ \mathrm{Tr}_{X/Y}\,.$$

Proposition 5.40. *The functional* $\mathrm{Tr}\,\omega$ *is a Weil differential on* X*, and we have*

$$\mathrm{div}(\mathrm{Tr}\,\omega) = f^*\,\mathrm{div}(\omega) + \mathcal{D}_{X/Y}$$

where $\mathcal{D}_{X/Y}$ *is the discriminant divisor.*

Proof. We will verify first that $\mathrm{Tr}\,\omega$ is a Weil differential, using our characterisation. First, $\mathrm{Tr}\,\omega$ vanishes on $K(X)$ since $\mathrm{Tr}_{X/Y}(K(X)) = K(Y)$, giving iii). Since the trace is the sum of local traces, we see that $\mathrm{Tr}\,\omega_Q$ vanishes on $\widehat{O}^{\mathfrak{q}}_{X,Q}$ whenever Q is unramified, and there exists some x with $v_Q(x) = -1$, and $v_P(\mathrm{Tr}_{Q/P}(x)) = -1$, giving i). By the finiteness of the different divisor, even when Q is ramified, $\mathrm{Tr}\,\omega$ will vanish on some finite power of $\mathfrak{q}$, giving ii). Finally, the local components of $\mathrm{Tr}\,\omega$ will be not identically zero since each local trace form is nondegenerate by seperability of our extension $K(X)/K(Y)$.

For the second claim, locally at Q/P we need to compute the lowest power of $\mathfrak{q}$ that $\mathrm{Tr}\,\omega$ vanishes identically on. First, assume that $v_P(\mathrm{div}(\omega)) = 0$. Then the Q component of $\mathrm{div}(\mathrm{Tr}\,\omega)$ is given by the maximal $n \in \mathbb{Z}$ such that

$$\mathrm{Tr}_{Q/P}(\pi^{-n}\widehat{O}^{\mathfrak{q}}_{X,Q}) \in \widehat{O}^{\mathfrak{p}}_{Y,P}$$

for π a uniformiser at Q. This is precisely our definiton of $v_Q(\mathcal{D}_{X/Y})$. Now for an arbitrary ω, note that $\pi^m \cdot \operatorname{Tr}\omega = \operatorname{Tr}(\pi^m\omega)$ by linearity, and $\operatorname{div}(\pi^m\omega) = \operatorname{div}(\pi^m) + \operatorname{div}(\omega)$. Since for a uniformiser γ at P, we have $v_Q(\gamma) = e_{Q/P}$, we see locally that

$$v_Q(\operatorname{Tr}\omega) = e_{Q/P}v_P(\omega) + v_Q(\mathcal{D}_{X/Y}).$$

In view of the definition of f^*, the result follows. □

Corollary 5.41 (Riemann-Hurwitz formula). *For a finite separable morphism $f : X \to Y$ of curves, of degree n, we have:*

$$2g_X - 2 = n(2g_Y - 2) + \deg_X(\mathcal{D}_{X/Y}).$$

Proof. We take the degree of the Weil differential $\operatorname{Tr}\omega$ in two different ways. On one hand, it is $2g_X - 2$ since it is a Weil differential on X. On the other, propositions 3.48 and 5.40 give the left side. □

Corollary 5.42. *If $f : X \to Y$ is a finite separable map of curves, then $g_X \geq g_Y$.*

Proof. Since $\mathcal{D}_{X/Y}$ is effective, $\deg_X(\mathcal{D}_{X/Y})$ is positive, as is n, giving the result. □

5.7 Residues and differential forms

For a curve X, we have noted that while the divisor class of K_X is canonical, the actual divisor K_X depends on our choice of Weil differential. In light of Theorem 5.34, we should look for a canonical invertible sheaf $\mathcal{F}$ such that $[\operatorname{div}(\mathcal{F})] = [K]$, with Ω_D canonically isomorphic to the global sections of $\mathcal{F} \otimes L(-D)$.

In this section, we will describe this invertible sheaf, and sketch how the classical proof of the Riemann-Roch theorem can be accomplished using it. For the proofs of the statements in this section, see the excellent exposition of [Ser88].

Definition 5.43 (Differentials). The module of differentials $\Omega_{R,k}$ over k of a k algebra R is the quotient of the free R module on the symbols $\mathrm{d}f$, for $f \in R$ subject to the relations

$$\begin{aligned} \mathrm{d}(f+g) &= \mathrm{d}f + \mathrm{d}g \\ \mathrm{d}(fg) &= f\,\mathrm{d}g + g\,\mathrm{d}f \\ \mathrm{d}\alpha &= 0 \end{aligned}$$

for $\alpha \in k$.

It can also be characterised by its universal property, for any R-module M, $\Omega_{R/k}$ represents the functor of k-derivations valued in M.

$$\hom_R(\Omega_{R,k}, M) \cong Der_k(R, M)$$

Definition 5.44 (Differentials on curves)**.** For a perfect field k, and a curve X/k, the modules $\Omega_{U,k}$ of k derivations on each affine open set U glue to an invertible sheaf Ω_X on X.

From here, we will assume that k is perfect. One should think of the sections of Ω_X over U as differential 1-forms on X, via the interpretation of smooth cotangent vector fields on a smooth manifold M being derivations on the ring $C^\infty(M)$.

We call elements of $\Omega_X(U)$ regular differentials on U, and elements of $\Omega_{X_{x_{gen}}}$ rational differentials, denoted Ω_X^r. Since Ω_X is invertible, this space Ω_X^r is one dimensional over $K(X)$, and this corresponds to our $J(X)$.

Because Ω_X is invertible, its stalk at each P is a free $O_{X,P}$ module of rank one, and it can be shown that if π is uniformiser of $O_{X,P}$, then $\mathrm{d}\pi$ is a basis of this stalk. By expanding in terms of this uniformiser, one may easily show that the completed local ring is isomorphic to the power series ring $\kappa(P)[[\pi]]$.

Definition 5.45. Any rational differential $\omega \in \Omega_X^r$ can be expanded as

$$\omega = f(\pi) \cdot \mathrm{d}\pi$$

in $(\Omega_X)_P^{\mathfrak{p}}$ where f is a formal power series in π, with coefficients in $\kappa(P)$. If c_{-1} is the coefficient of π^{-1} in f, then we define the residue of w at P to be

$$\mathrm{Res}_P(\omega) := \mathrm{Tr}_{\kappa(P)/k}(c_{-1}).$$

Similarly, we define the order of ω at P to be $v_P(f)$.

One may show without much difficulty that the order of ω at P is independent of the uniformising parameter π. With significant work, see [Ser88], one may also show that the residue is also independent of this choice. In the complex geometry situation, this is precisely the usual residue of a meromorphic differential at a point P, the invariance of which can be proved via contour integration, see [Mir95].

Similar to the complex case, one has the following theorem, though as expected, it is more difficult to prove in this algebraic setting.

Theorem 5.46 (The residue theorem)**.** *For any $\omega \in \Omega^r_X$, we have*

$$\sum_{P \in X} \mathrm{Res}_P(\omega) = 0.$$

In this case of perfect base field k, this allows us to canonically identify $J(X)$ and Ω^r_X.

Theorem 5.47. *For X a curve over a perfect field k, we have a canonical isomorphism of $K(X)$ vector spaces*

$$\Omega^r_X \xrightarrow{\tau} J(X)$$

between rational differentials and Weil differentials on X.

For any rational differential ω, τ is given by

$$\tau(\omega)((\alpha_P)) = \sum_{P \in X} \mathrm{Res}_P(\alpha_P \cdot \omega).$$

From this, we may express the duality of Theorem 5.34 coordinate independently as the perfect pairing

$$H^0(\Omega_X \otimes L(-D)) \otimes_k H^1(D) \to k.$$

Remark 5.48. This comparison does not strictly require a perfect field k, the sheaf of differentials will behave similarly for any curve with a separable morphism to $\mathbb{P}^1_k$. Over non-perfect fields however, not all curves are of this form. In this somewhat pathological setting, the author does not know of a similar canonical invertible sheaf giving rise to the divisor class K_X.

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